WEB-BASED EDUCATION SYSTEM FOR ELEMENTARY MATHEMATICS

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ABSTRACT

Elementary mathematics, for instance, algebraic equations is an important field of study in higher education. In particular, graphical representations of curves of equations make the understanding of mathematical problem easy for students. Therefore, we have developed a web-based education system with a graphics database for elementary mathematics, particularly, algebraic equations. In this paper, the analytical properties of quadratic and cubic algebraic equations are discussed on the basis of the Sturm concept because of its educational value and usefulness in analyzing and designing a real system. Some graphical representations utilized for the web-based education system of algebraic equations are shown. An intranet e-learning system constructed in our laboratory on a Linux web-server (Apache HTTP server) is presented.

Keywords: Intranet e-learning systems, Apache web server, algebraic equations, Sturm theorem, division algorithm, interval systems

1 INTRODUCTION

Elementary mathematics, for instance, the analytic properties of algebraic equations is an important field of study in higher education. In particular, graphical representations of these curves make the understanding of mathematical problems easy for students. In general, theoretical problems in physical, biological and economical systems may require algebraic equations to be solved. However, the meaning of the solution of an algebraic equation has changed with the rapid improvement in computer performance. Usually, the roots of an algebraic equation are numerically calculated. Therefore, students (and also researchers) are not interested in the analytical properties of an algebraic (nonlinear) equation. Hence, in this paper, a division algorithm based on the classic Sturm theorem [1, 2, 3] is reconsidered as an analytical method. The method of determining the number of roots in a specified range and area is not only important from an academic point of view but also useful for analyzing and designing a real system with some uncertainty. The set of roots (infinite set) of algebraic equations with uncertain (interval set) coefficients [4] cannot be determined numerically by finite calculations.

Hence, we have developed a web-based education system with a graphics database for algebraic equations. In this paper, the analytical properties of quadratic and cubic algebraic equations are discussed on the basis of Sturm concept because of its educational value and usefulness in analyzing and designing a real system [5]. Some graphical representations utilized for the web-based education for algebraic equations are shown. An intranet e-learning system constructed in our laboratory on a Linux web-server (Apache HTTP server) is presented.

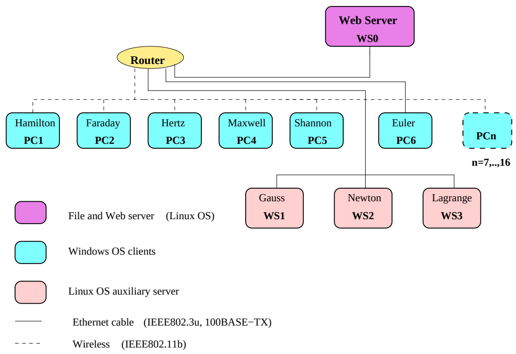

Fig. 1. Computer network.

2 INTRANET E-LEARNING SYSTEM

Figure 1 shows the network topology of the intranet e-learning system that was used in this study. The computers(hosts) are connected by either Ethernet cables or wireless methods. 'Hamilton'(PC1), 'Faraday'(PC2), etc., are our names for these computers, which have a Windows operating system. In this type of small-scale wired/wireless network, 16 client PCs can be connected. In the figure, dotted lines denote wireless connections and the bold lines are wired connections. Workstation(W0) is the main Linux Apache HTTP web server and also Samba file server. Workstations WS1, WS2 and WS3 are Linux web and file servers in order to assist the main server.

Figure 2 shows the entire intranet system of our laboratory. Pages Research 1 and Research 2 contain our literatures on discretized control and interval systems. As an example, e-Learning 1 (that is, Mathematics Courseware, Elementary Mathematics, Algebraic Equations, Quadratic Equations) is presented in this paper.

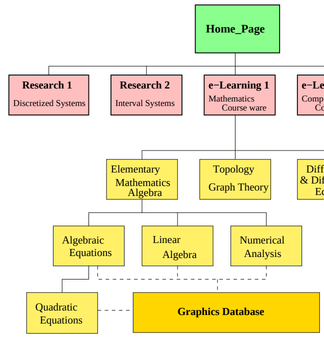

Fig. 2. Intranet e-learning and management system.

3 ALGEBRAIC EQUATIONS, GRAPHICS DATABASE

Each of the e-Learning systems is linked to Graphics Database as shown in Fig. 2. The curve that is relevant to the problem is selected from this database. The following algebraic equations have been discussed based on the Sturm concept.

3.1 Real Roots Problem

In this paper, we treat quadratic and cubic equations because of their simplicity. Of course, the procedure described in this section can be applied to higher order (general) algebraic equations. First, we consider a problem to determine the condition under which two real roots of the quadratic equation

$$f(x) = ax^2 + bx + c = 0 \tag{1}$$

exist in the interval, $x_1 < x \leq x_2$. An elementary approach to this problem is given as follows. From the axis of parabola (i.e., $\bar{x} = -b/2a$), the following condition must be satisfied:

$$x_1 < \bar{x} \leq x_2. \tag{2}$$

Moreover, the condition can be written as:

$$f(x_1) = ax_1^2 + bx_1 + c > 0, \tag{3}$$
$$f(x_2) = ax_2^2 + bx_2 + c > 0, \tag{4}$$
$$D = b^2 - 4ac > 0. \tag{5}$$

On the other hand, the classic Sturm theorem asserts the following. For $f_0(x) = f(x) = ax^2 + bx + c$ and $f_1(x) = f'(x) = \dfrac{df(x)}{dx} = 2ax+b$, the following division algorithm can be applied:

$$f_0(x) = q_1(x)f_1(x) - f_2. \tag{6}$$

The results of (6) are easily obtained as follows:

$$q_1(x) = \frac{2ax+b}{4a}, \qquad f_2 = \frac{b^2-4ac}{4a}.$$

The Sturm theorem states that if the number of sign changes of the function series $f_0(x),\ f_1(x),\ f_2(x)$ at $x =$ x_1 and $x = x_2$ are defined as $V(x_1)$ and $V(x_2)$ respectively, the following condition must be satisfied:

$$V(x_1) - V(x_2) = \mathbf{2}. \tag{7}$$

Here, **2** is the number of roots that exist in the interval.

From the above discussion, the following conditions are derived when $a > 0$:

$$f_0(x_1) > 0,\ f_1(x_1) < 0,\ f_2 > 0, \tag{8}$$
$$f_0(x_2) > 0,\ f_1(x_2) > 0,\ f_2 > 0. \tag{9}$$

Obviously, these inequalities correspond to (2), (3), (4) and (5). (When $x_1 \to -\infty$ and $x_2 \to +\infty$, the necessary and sufficient condition becomes $f_2 > 0$.) Therefore, $V(x_1) - V(x_2) = \mathbf{2}$ was confirmed. Figure 3 shows $f_0(x)$, $f_1(x)$ and f_2 where (8) and (9) are satisfied. In this study, such a figure is prepared in the graphics database.

On the contrary, when the following inequalities are given:

$$f_0(x_1) < 0,\ f_1(x_1) > 0,\ f_2 < 0, \tag{10}$$
$$f_0(x_2) < 0,\ f_1(x_2) < 0,\ f_2 < 0, \tag{11}$$

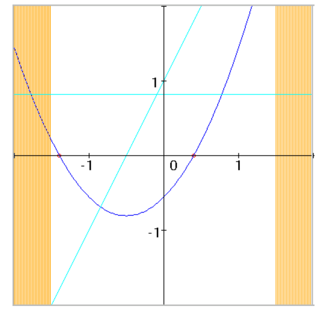

Fig. 3. Quadratic curve, roots and a permitted range ($f(x) = x^2 + x - 0.56$, $-1.5 < \bar{x} \leq 1.2$).

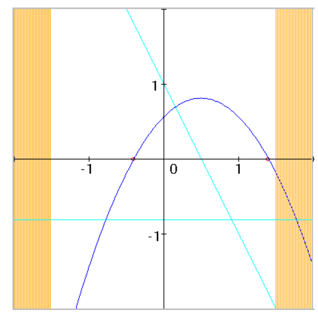

Fig. 4. Quadratic curve, roots and a permitted range ($f(x) = -x^2 + x + 0.56$, $-1.2 < \bar{x} \leq 1.5$).

a figure as shown in Fig. 4 is picked out. This is the case where $a < 0$. Of course, when the number of roots that exist in the interval is $N = 1$ (or $N = 0$), the conditions corresponding to (8) and (9) can be obtained in the same way. In the case of $N = 1$, the following inequalities can

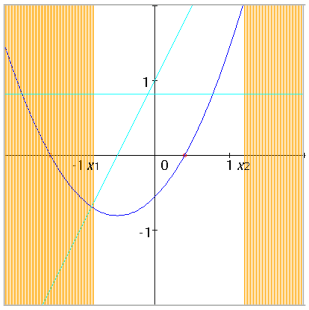

Fig. 5. Quadratic curve, roots and a permitted range ($f(x) = x^2 + x - 0.56$, $-0.8 < \bar{x} \leq 1.2$).

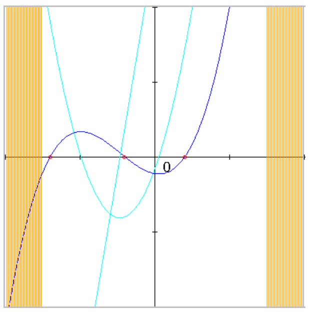

Fig. 6. Cubic curve, roots and a permitted range ($f(x) = x^3 + 1.4x^2 - 0.16x - 0.224$, $-1.5 < \bar{x} \leq 1.2$).

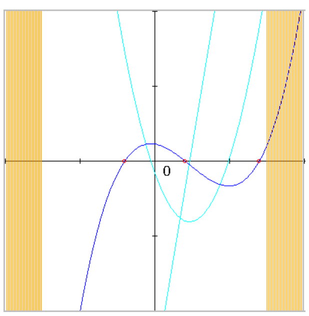

Fig. 7. Cubic curve, roots and a permitted range ($f(x) = x^3 + 1.4x^2 - 0.16x - 0.224$, $-0.8 < \bar{x} \leq 1.2$).

be obtained:

$$f_0(x_1) < 0,\ f_1(x_1) < 0,\ f_2 > 0, \tag{12}$$
$$f_0(x_2) > 0,\ f_1(x_2) > 0,\ f_2 > 0. \tag{13}$$

Hence, $V(x_1) - V(x_2) = \mathbf{1}$ was confirmed. Figure 5 shows the selected figure for (12) and (13). Such a figure is also stored in the graphics database.

Figure 6 shows an example of a cubic equation with three real roots. The division algorithm for a cubic equation, which corresponds to (6), is given as follows:

$$\begin{aligned} f_0(x) &= q_1(x)f_1(x) - f_2(x), \\ f_1(x) &= q_2(x)f_2(x) - f_3. \end{aligned} \tag{14}$$

As for an example shown in Fig. 6, the following inequalities can be obtained:

$$\begin{aligned} &f_0(x_1) < 0,\ \ f_1(x_1) > 0,\ \ f_2(x_1) < 0,\ \ f_3 > 0, \\ &f_0(x_2) > 0,\ \ f_1(x_2) > 0,\ \ f_2(x_2) > 0,\ \ f_3 > 0. \end{aligned}$$

Thus, $V(x_1) - V(x_2) = \mathbf{3}$ is confirmed. When the interval considered here is as shown in Fig. 7, the following inequalities can be given:

$$\begin{aligned} &f_0(x_1) > 0,\ \ f_1(x_1) < 0,\ \ f_2(x_1) < 0,\ \ f_3 > 0, \\ &f_0(x_2) > 0,\ \ f_1(x_2) > 0,\ \ f_2(x_2) > 0,\ \ f_3 > 0. \end{aligned}$$

Clearly, $V(x_1) - V(x_2) = \mathbf{2}$ is confirmed.

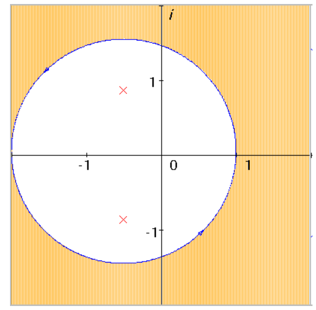

Fig. 8. Complex plane and a circular contour ($f(z) = z^2 + z + 1$, $u_0 = -0.5$, $v_0 = 0.1$, $\rho = 1.5$).

3.2 Complex Roots Problem

The result described in **3.1** can be extended to a complex roots problem. For a quadratic equation,

$$f(z) = az^2 + bz + c = 0, \tag{15}$$

if there is no root in the whole area of real values, for example, the following inequalities are given:

$$\begin{aligned} &f_0(-\infty) > 0,\ \ f_1(-\infty) < 0,\ \ f_2 < 0, \\ &f_0(+\infty) > 0,\ \ f_1(+\infty) > 0,\ \ f_2 < 0, \end{aligned}$$

(i.e., $V(-\infty) - V(+\infty) = \mathbf{0}$), complex roots should be considered as:

$$\bar{z} = \bar{u} + i\bar{v} = -\frac{b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a}, \tag{16}$$

where $i = \sqrt{-1}$. For a cubic equation,

$$f(z) = az^3 + bz^2 + cz + d = 0, \tag{17}$$

if there exists only one root in the whole area of real values, for example, the following inequalities are given:

$$\begin{aligned} &f_0(-\infty) < 0,\ \ f_1(-\infty) > 0,\ \ f_2(-\infty) < 0,\ \ f_3 < 0, \\ &f_0(+\infty) > 0,\ \ f_1(+\infty) > 0,\ \ f_2(+\infty) > 0,\ \ f_3 < 0, \end{aligned}$$

(i.e., $V(-\infty) - V(+\infty) = \mathbf{1}$), complex roots should be considered as well.

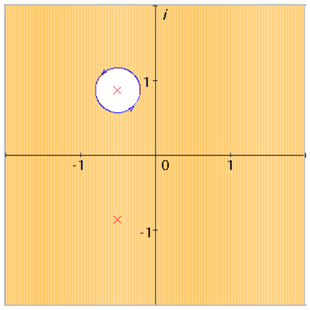

Fig. 9. Circular contour that encircles only one root ($f(z) = z^2 + z + 1$, $u_0 = -0.5$, $v_0 = 0.866$, $\rho = 0.3$).

Here, we define a complex plane $z = u + iv$ as shown in Fig. 9, and simply consider the following circular contour [6, 7]:

$$z = \rho e^{i\theta} + u_0 + iv_0, \quad (\theta : -\pi \to \pi), \tag{18}$$

where ρ, (u_0, v_0) and θ are the radius, the center and the angle of rotation of a specified circle, respectively. (The problem of a sectorial area in the complex plane was discussed in the previous paper [8].) Figure 9 shows an example of the circular contour that encircles only one root. The circular contour (18) can also be written by the following rational function of a real variable $\nu = \tan\theta/2$:

$$z = \rho \cdot \frac{1 + i\nu}{1 - i\nu} + u_0 + iv_0 = \frac{\alpha(\nu) + i\beta(\nu)}{1 - i\nu}, \tag{19}$$

where $\alpha(\nu) = \rho + u_0 + v_0\nu$ and $\beta(\nu) = v_0 + (\rho - u_0)\nu$.

(1) Quadratic Equation When the quadratic equation (15) is considered, the following expression for a complex function is obtained:

$$\Phi(i\nu) = (1 - i\nu)^2 f(i\nu) = P(\nu) + iQ(\nu). \tag{20}$$

By using notations, $f_0(\nu) = P(\nu)$, $f_1(\nu) = Q(\nu)$, the real and imaginary parts of the complex function can be given as:

$$\begin{aligned} f_0(\nu) &= a(\alpha^2(\nu) - \beta^2(\nu)) + b\alpha(\nu) + c \\ &= a_0\nu^2 + b_0\nu + c_0, \qquad (21) \\ f_1(\nu) &= 2a\alpha(\nu)\beta(\nu) + b\beta(\nu) \\ &= a_1\nu^2 + b_1\nu + c_1. \qquad (22) \end{aligned}$$

Based on the extended Sturm theorem, when the division algorithm for a quadratic equation with complex coefficients,

$$\begin{aligned} f_0(\nu) &= q_1(\nu)f_1(\nu) - f_2(\nu), \\ f_1(\nu) &= q_2(\nu)f_2(\nu) - f_3(\nu), \\ f_2(\nu) &= q_3(\nu)f_3(\nu) - f_4, \end{aligned} \tag{23}$$

is applied, a series of polnomials,

$$\begin{aligned} f_2(\nu) &= b_2\nu + c_2, \\ f_3(\nu) &= b_3\nu + c_3, \\ f_4 &= c_4. \end{aligned} \tag{24}$$

are obtained.

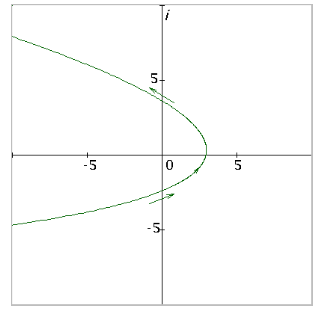

Fig. 10. Mapping $\Phi(i\nu)$ for the specified circle in Fig. 8.

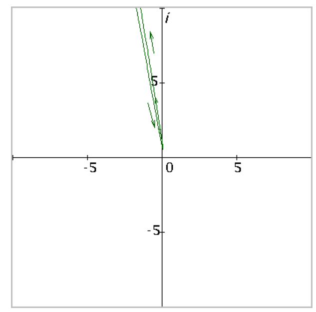

Fig. 11. Mapping $\Phi(i\nu)$ for the specified circle in Fig. 9.

When the specified contour encircles two roots of the quadratic equation (e.g., as shown in Fig. 8), the argument change of $\Phi(i\nu)$ becomes $4\pi - 2\pi = 2\pi$ by adding a change of -2π to the argument of $(1 - i\nu)^2$. Figure 10 shows Φ curve of such a case. Therefore, when the contour encircles only one root (e.g., as shown in Fig. 9), the argument change of $\Phi(i\nu)$ becomes $2\pi - 2\pi = 0$. That is, the vector locus does not turn around the origin as shown in Fig. 11.

The extended Sturm theorem states that if the number of sign changes of $f_0(\nu)/f_1(\nu)$ that crosses zero (from $-$ to $+$) when $\nu : -\infty \to +\infty$ is expressed as $N(-\infty, +\infty)$ and the number of sign changes of function series $f_0(\nu)$, $f_1(\nu)$, $f_2(\nu)$, $f_3(\nu)$, and f_4 is expressed as $V(\nu)$, the following relationship holds[6, 7]:

$$N(-\infty, +\infty) = V(-\infty) - V(+\infty) = n - 2\mu, \tag{25}$$

where n is the order of the equation and μ is the number of encircled roots. Therefore, when the specified contour encircles two roots, the following result is given:

$$V(-\infty) - V(+\infty) = 2 - 2 \cdot \mathbf{2} = -2. \tag{26}$$

When the contour encircles one root,

$$V(-\infty) - V(+\infty) = 2 - 2 \cdot \mathbf{1} = 0. \tag{27}$$

is obtained. The condition of (26) (or (27) for one root) results in checking whether the following ratios (ratios of for the highest order coefficients of the polynomials) are negative or not:

$$\lim_{\nu \to +\infty} \frac{f_1(\nu)}{|\nu| f_2(\nu)} = \frac{a_1}{b_2}, \quad \lim_{\nu \to +\infty} \frac{f_3(\nu)}{|\nu| f_4} = \frac{b_3}{c_4}. \tag{28}$$

The result is given as follows:

$$\frac{a_1}{b2} = \frac{0.3}{-30.1}, \quad \frac{b_3}{c_4} = \frac{-2.96}{2.93},$$

when the specified circle is as shown in Fig. 8. That is, $\mu = 2$ is obtained. Moreover, the following is given:

$$\frac{a_1}{b2} = \frac{0.52}{-0.03}, \quad \frac{b_3}{c_4} = \frac{2.82}{0.19},$$

when the specified circle is as shown in Fig. 9. That is, $\mu = 1$ is obtained. Figure 11 show the mapping curve of $\Phi(i\nu)$ for the specified circle.

(2) Cubic Equation The above discussion on quadratic equations can be extended to the problem on cubic and also general algebraic equations. As for cubic equation (17), the following complex function is given:

$$\Phi(i\nu) = (1 - i\nu)^3 f(i\nu) = P(\nu) + iQ(\nu). \quad (29)$$

By using the same notations, $f_0(\nu) = P(\nu)$, $f_1(\nu) = Q(\nu)$, the following polynomials can be obtained:

$$f_0(\nu) = a_0\nu^3 + b_0\nu^2 + c_0\nu + d_0, \quad (30)$$
$$f_1(\nu) = a_1\nu^3 + b_1\nu^2 + c_1\nu + d_1. \quad (31)$$

From (29), when the circular contour encircles 3 roots of the quadratic equation, the argument change of $\Phi(i\nu)$ becomes $6\pi - 3\pi = 3\pi$ by adding a change of -3π to the argument of $(1 - i\nu)^3$. Therefore, when the contour encircles only one root, the argument change of $\Phi(i\nu)$ becomes $-\pi$.

By applying the division algorithm as shown in (23), a series of polynomials,

$$\begin{aligned} f_2(\nu) &= b_2\nu^2 + c_2\nu + d_2, \\ f_3(\nu) &= b_3\nu^2 + c_3\nu + d_3, \\ f_4(\nu) &= c_4\nu + d_4, \\ f_5(\nu) &= c_5\nu + d_5, \\ f_6 &= d_6. \end{aligned} \quad (32)$$

are obtained. From the highest order coefficients of these polynomials, the number of encircled roots can be determined by checking whether the following ratios are negative or not:

$$\lim_{\nu\to+\infty} \frac{f_1(\nu)}{|\nu| f_2(\nu)} = \frac{a_1}{b_2}, \quad \lim_{\nu\to+\infty} \frac{f_3(\nu)}{|\nu| f_4(\nu)} = \frac{b_3}{c_4},$$
$$\lim_{\nu\to+\infty} \frac{f_5(\nu)}{|\nu| f_6} = \frac{c_5}{d_6}. \quad (33)$$

The result is given as follows:

$$\frac{a_1}{b2} = \frac{-2.96}{9.15}, \quad \frac{b_3}{c_4} = \frac{-1.29}{48.7}, \quad \frac{c_5}{d_6} = \frac{6.87}{-5.75},$$

when the specified circle is as shown in Fig. 12. That is, $\mu = 3$ is obtained. Figure 13 show the mapping curve of $\Phi(i\nu)$ for the specified circle.

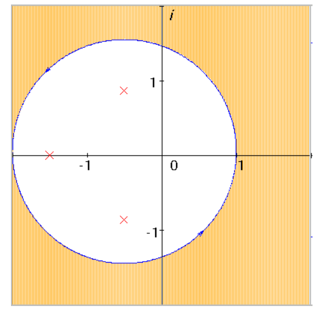

Fig. 12. Cubic equation and a specified circle ($f(z) = z^3 + 2z^2 + 2z + 1$, $u_0 = -0.5$, $v_0 = 0.1$, $\rho = 1.5$).

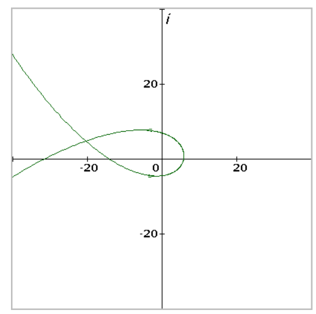

Fig. 13. Mapping $\Phi(i\nu)$ for the specified circle.

4 INTERVAL SYSTEMS

Many real systems with uncertainty data (or parameters) exist in the physical, biological and economical fields. These uncertainties should be considered in some bounded area [4, 6]. In other words, the following equation should be treated instead of, e.g., (17):

$$f(z) = [a^-, a^+]z^3 + [b^-, b^+]z^2 + [c^-, c^+]z + [d^-, d^+] = 0. \quad (34)$$

Here, $a^-,\ b^-, \cdots$ and $a^+,\ b^+, \cdots$ are lower and upper bounds of the coefficients, respectively. Each of the coefficients is expressed as an interval set. Therefore, the solution of (34) should be considered to be some bounded area. When considering a circular area in the complex plane as given in (18) and (19), a set of vector loci (29) can be covered by rectangles with the following four corners: (P^+, iQ^+), (P^+, iQ^-), (P^-, iQ^-), (P^-, iQ^+) [7]. In this case, each extreme point can be expressed as a polynomial for variable ν with complex coefficients. Therefore, if the condition given in (28) and (33) is satisfied with respect to these four polynomials, the number of roots in the specified circular area does not change. It is a sufficient condition in which the interval system has the specified robust performance.

Figure 14 shows a case where a circular contour encircles three roots of a cubic equation with uncertain coefficients. (The areas of roots are also plotted in the figure.) Polytopes mapping of the interval (complex) function $f(i\nu)$ is calculated as shown in Fig. 15. Figures 16 and 17 are polytopic mappings of the interval (complex) functions

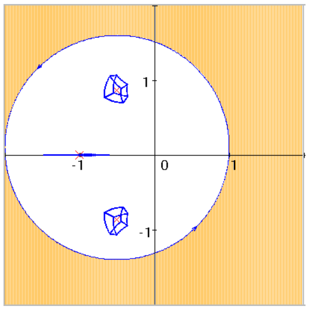

Fig. 14. Cubic equation and a specified circle ($f(z) = z^3 + [1.8, 2.2]z^2 + [1.8, 2.2]z + [0.9, 1.1]$, $u_0 = -0.5$, $v_0 = 0.1$, $\rho = 1.5$).

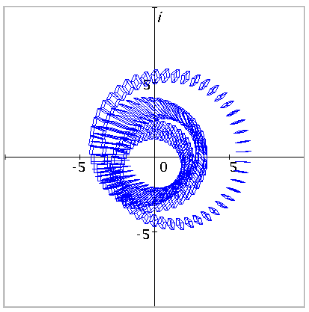

Fig. 15. Polytopes mapping $f(i\nu)$ for the specified circle in Fig. 14.

$\Phi(i\nu)$, respectively. As shown in these figures, the polytopic mappings and other curves can be zoomed in and out by using an appropriate ghost-view window.

5 CONCLUSIONS

In this paper, we have reported a web-based education system with a graphics database for solving algebraic equations. The analytical properties of quadratic and cubic algebraic equations were discussed based on the Sturm concept because of its educational value and usefulness in analyzing and designing a real system with uncertainty. Some graphical representations utilized for the web-based education of algebraic equations were shown. The problem of an algebraic equation with interval coefficients was also discussed. In addition, as an example of a small-scale network, an intranet e-learning system on a Linux web-server (Apache HTTP server) was presented.

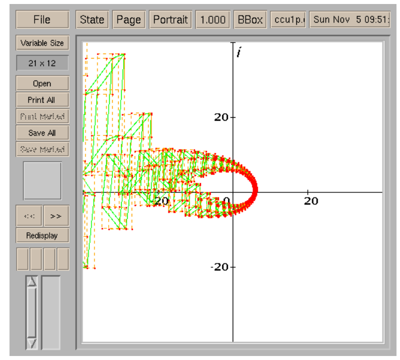

Fig. 16. Polytopes mapping of $\Phi(i\nu)$ for a circular contour as shown in Fig. 14.

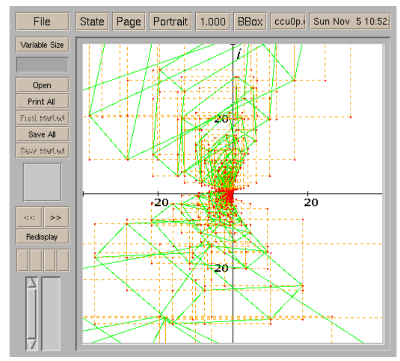

Fig. 17. Polytopes mapping of $\Phi(i\nu)$ for a circular contour as shown in Fig. 9.

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