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Stability and Security Analyses of Event-Driven Discrete Control Systems Based on Lattice Theory

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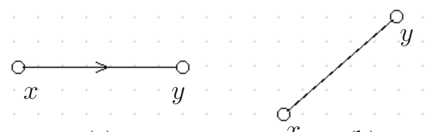
Abstract: Nowadays there are many event-driven types of discrete control systems in practice, e.g., manufacturing automation systems, industrial and welfare robots, networked control systems, and so forth. Therefore, in this paper, the stability and (especially) security problems of event-driven control systems are considered. First, the lattice concept based on ordered sets is reviewed in general. Then, integer lattice coordinates are introduced, and the relative stability and boundedness (i.e., security) of event-driven discrete systems is investigated using multiple metrics and simultaneous linear inequalities. Numeric examples of qualitative and quantitative problems in the three-dimensional space are given to express visually and to clarify the stability and security of event-driven control systems.

Keywords: Discrete event systems; lattice coordinates; multiple metrics; inequality conditions; relative stability

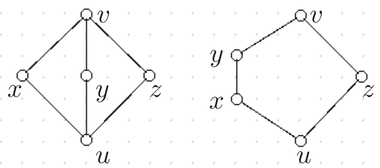
1. INTRODUCTION

At present, there are many event-driven types of discrete systems in practice, e.g., manufacturing systems, industrial and welfare robots, networked control systems (e.g., popularly known as IoT), and so on. Therefore, in this paper, the (finite-time) stability of event-driven (in other words, event-based) control systems is studied. First, the lattice concept based on ordered sets is reviewed in general [1-4]. Then, integer lattice coordinates are introduced, and the stability and security of event-driven discrete control systems is analyzed using multiple metrics and simultaneous linear inequalities. Numerical examples in the three-dimensional space (because of the visual expression) will be given to clarify the relative stability and boundedness [5] (i.e., security) of event-driven control systems.



(a) (b)

Fig. 1 Directed graph and Hasse diagram with two points.



(a) $\mathbf{M}_3$ (b) $\mathbf{N}_5$

Fig. 2 Non-distributive sublattices.

2. ORDERED SETS AND LATTICES

A lattice is an ordered set E in which every pair of elements (and hence every finite subset) has an infimum (meet, $\wedge$) and a supremum (join, $\vee$). Thus, we often denote a lattice by $(L; \wedge, \vee)$, $(L; \wedge, \vee, \leq)$ or $(L; \wedge, \vee, \preceq)$. Here, $\leq$ (or $\geq$) is a quantitative relation and $\preceq$ (or $\succeq$) is a qualitative relation, and the former is read "less than or equal" (or "greater than or equal") and the latter is read, for example, "precedes" (or "succeeds") [6].

Covering relation and graph. In an ordered set $(E; \leq)$, we say that x is covered by y (or that y covers x) if $x < y$ and there is no $z \in E$ such that $x < z < y$. We denote this by using the notation $x \sqsubset y$ Thus, points x and y that satisfy $x \sqsubset y$ are also called *adjacent*.

The adjacent points $x \sqsubset y$ can be represented by using a diredted graph as shown in Fig. 1 (a). That is, we join the points (vertices) representing x and y by a line segment with an arrow (directed edge). However, instead of drawing an arrow from x to y, we sometimes place y higher than x and draw a line (without a arrow) between them as shown in Fig. 1 (b). It is then understood that an upward movement indicates succession (otherwise precession). Such a graphical representation is referred to as a *Hasse diagram*.

Distributive and Modular laws. In the lattice theory, [1] the following propositions are important:

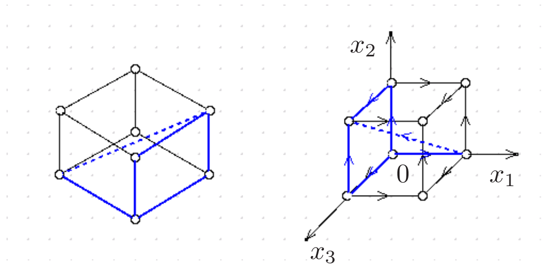
(1) A lattice L is said to be *distributive* if it satisfies
$(\forall x, y, z \in L)\ \ z \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

(2) A lattice L is said to be *modular* if it satisfies
$(\forall x, y, z \in L)\ \ x \geq z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z$.

Although the modular law (2) can be derived from (1), the converse is not always valid. Figures 2 (a) and (b) show sublattices which are not distributive. Here, $\mathbf{M}_3$ is modular, however, it is not distributive. On the other

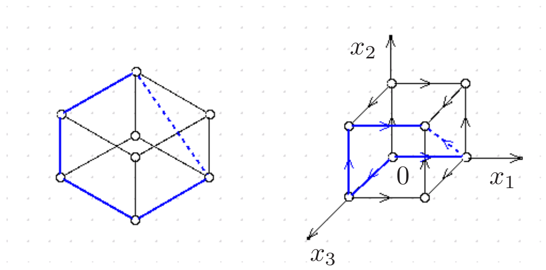
† Yoshifumi Okuyama is the presenter of this paper.

[1] The original meaning of a "lattice" is a structure of cross laths with interstices serving as screen, door, etc (from C.O.D.). It corresponds to a "koushi' in Japanese traditional house. The word lattice in the mathematical terms is translated into Japanese as "soku" or "koushi". It should be noted that the former is with respect to **Set Theory and Topology**[4] and the latter is with respect to **Geometry of Numbers**[7] (especially, in a two-dimensional plane). In this paper, both concepts will be introduced.



(a) (b)

Fig. 3 Cubic and $\mathbf{M}_3$ lattices by Hasse diagram, and directed graphs in the 3D coordinates.



(a) (b)

Fig. 4 Cubic and $\mathbf{N}_5$ lattices by Hasse diagram, and directed graphs in the 3D coordinates.

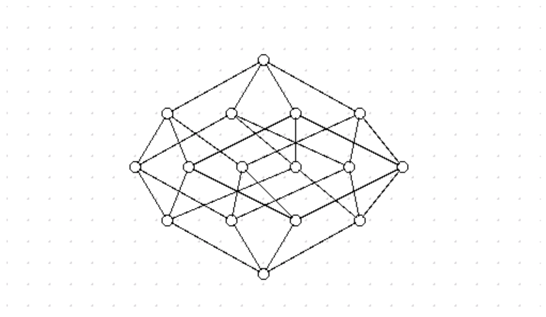


Fig. 5 Hasse diagram for a 4D lattice.

hand, $\mathbf{N}_5$ is not modular and also not distributive. In the literatures[2, 4], it is said that a lattice is distributive if and only if it has no sublattice of either of these $\mathbf{M}_3$ and $\mathbf{N}_5$ in Fig. 2.

Figures 3 and 4 show Hasse diagrams and directed graphs of cubic lattices in the 3D coordinates [2]. Here, the 'blue' lines denote non-distributive (forbidden) lattices $\mathbf{M}_3$ and $\mathbf{N}_5$. In either of the diagrams, if the dotted line (edge) does not exist, the lattice will be unified into a cubic lattice in the 3D orthogonal coordinates. In general, any sublattice (without such dotted edges) satisfies the above properties (1) and (2). Therefore, it can be confirmed that a lattice in the usual (quantitative and parallel) coordinates is *distributive* and *modular*. When we consider a four-dimensional cube, the Hasse diagram is given as shown in Fig. 5.

[2] In the diagrams and in the following examples, the discrete systems are restricted in a 3D space, because the behavior of them can be expressed visually. Of course, the ideas will be extended in the multi-dimensional space.

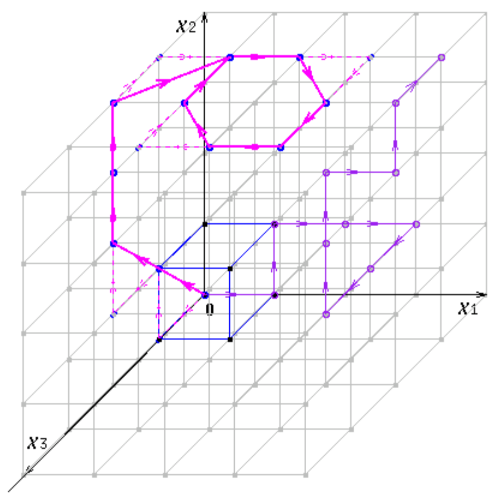


Fig. 6 3D orthogonal lattice coordinates and state traces.

3. LATTICE COORDINATES

Based on the above premise, let us consider a lattice constructed by qualitative ordered sets (e.g., events or states), $x^0 \prec x^1 \prec x^2 \prec \cdots$, where the relation $\prec$ is a symbol that is read "strictly precedes". Obviously, the ordered sets can be corresponded to quantitative numbers. If the relations are covering ones (i.e., *adjacent*), the ordered sets can be replaced by simply integer numbers as follows (i.e., a *chain*):

$$\begin{array}{ccccccc} x_i^0 & \sqsubset & x_i^1 & \sqsubset & x_i^2 \sqsubset & \cdots & \sqsubset & x_i^N \\ \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ 0 & & 1 & & 2 & \cdots & & N \end{array} \quad (i = 1, 2, \cdots, n) \qquad (1)$$

With respect to an ordered set of dimension three[3],event-series (or state-traces) will be given in a 3D space. Figure 6 shows an expression of the 3D lattice coordinates. Here, a box drawn by 'blue' lines corresponds to *Boolean lattice*[3].

The following example of (adjacent points) state-traces is drawn by 'purple' lines in the figure.

$$\begin{array}{ccccccc} (x_1^0, x_2^0, x_3^0) & \sqsubset & (x_1^1, x_2^1, x_3^1) & \sqsubset & (x_1^2, x_2^2, x_3^2) & \cdots \\ \downarrow & & \downarrow & & \downarrow & \\ (0,0,0) & & (1,0,0) & & (2,0,0) & \cdots \end{array}$$

$$\begin{array}{ccccc} \cdots & \sqsubset & (x_1^{11}, x_2^{11}, x_3^{11}) & \sqsubset & (x_1^{12}, x_2^{12}, x_3^{12}) \\ & & \downarrow & & \downarrow \\ \cdots & & (4,3,4) & & (4,4,4) \end{array} \qquad (2)$$

These lines and vertices construct the lattice coordinates. In Fig. 6, traces (1) and (2 are restricted only to the first quadrant (i.e., $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$). Of course, it can be expanded to the quadrants with negative numbers.

An example of state-trace (finally periodic trace),

$$\begin{array}{cccccc} (x_1^0, x_2^0, x_3^0) & \prec & (x_1^1, x_2^1, x_3^1) & \prec & (x_1^2, x_2^2, x_3^2) & \prec \cdots \\ \downarrow & & \downarrow & & \downarrow & \\ (0,0,0) & & (0,1,1) & & (0,2,2) & \cdots \end{array}$$

[3] The conccept is based on [4]

$$\begin{array}{ccccc} \cdots (x_1^8, x_2^8, x_3^8) & \prec & (x_1^9, x_2^9, x_3^9) & \sqsubset & (x_1^{10}, x_2^{10}, x_3^{10}) \\ \downarrow & & \downarrow & & \downarrow \\ \cdots\ (2,4,3) & & (1,4,2) & & (1,4,1) \end{array}$$

is drawn with arrows by 'magenta' lines as shown in Fig. 6. Note that in this case the edges do not always connect adjacent vertices. If the state traces adjacent vertices, the edges will be drawn by 'pink' lines.

4. EVENT-DRIVEN DISCRETE SYSTEMS AND STATE TRACES

In general, event-driven types of discrete systems can be written as

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{f}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)), \quad k = 0, 1, 2, \ldots,$$
$$\boldsymbol{x}(t_k) \in \mathbb{Z}^n, \quad \boldsymbol{f} : \mathbb{Z}^n \times \mathbb{Z}^m \to \mathbb{Z}^n.$$

In order to study the relative stability problem, we consider the following semi-linear discrete systems.

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{\Phi}(t_{k+1}, t_k)\boldsymbol{x}(t_k) + \boldsymbol{f}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)). \quad (3)$$

It is assumed that the transition matrix $\boldsymbol{\Phi}(\cdot, \cdot)$ is considered time-invariant and written as

$$\boldsymbol{\mathcal{P}} := \boldsymbol{\Phi}(t_{k+1}, t_k) \in \mathbb{Z}^{n \times n}, \quad \forall k \in \mathbb{N}. \quad (4)$$

Recurrent Systems and 0-1 Matrices If we simplify it only with respect to the system structure, the transition matrix may be written as:

$$\boldsymbol{\mathcal{P}} \in \mathbb{I}^{n \times n} \subseteq \mathbb{Z}^{n \times n}, \ \mathbb{I} := \{-1, 0, 1\}. \quad (5)$$

When considering nonnegative entries, we will define the following structure matrix [8, 9]:

$$\boldsymbol{\mathcal{P}} \in {\mathbb{I}_+}^{n \times n} \subseteq \mathbb{Z}^{n \times n}, \ \mathbb{I}_+ := \{0, 1\}. \quad (6)$$

A matrix each of whose entries is either 0 or 1 is called a (0,1)-matrix[9]. As for third-order periodic systems, the (0,1)-matrices are given by

$$\boldsymbol{\mathcal{P}}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{\mathcal{P}}_3^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

and the directed graphs are as shown in Fig. 7 [4]. With respect to fourth-order periodic systems, the (0,1)-matrices are given by

$$\boldsymbol{\mathcal{P}}_{41} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \boldsymbol{\mathcal{P}}_{42} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \boldsymbol{\mathcal{P}}_{43} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

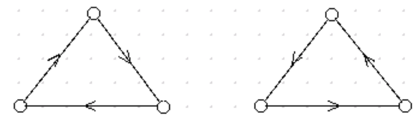
$$\boldsymbol{\mathcal{P}}_{41}^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{\mathcal{P}}_{42}^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \boldsymbol{\mathcal{P}}_{43}^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and the directed graphs are as shown in Figs. 8 and 9.

By expanding (3), the following equation is obtained:

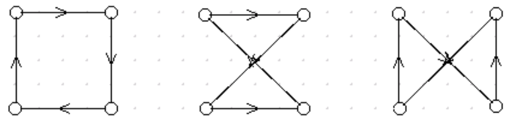
$$\boldsymbol{x}(t_k) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0) + \sum_{l=1}^{k} \boldsymbol{\mathcal{P}}^{k-l} \boldsymbol{f}(\boldsymbol{x}(t_{l-1}, \boldsymbol{e}(t_{l-1}))). \quad (7)$$

[4] It should be noted that the relationship between a directed graph and a matrix expression corresponds to $\boldsymbol{\mathcal{P}}^T$.



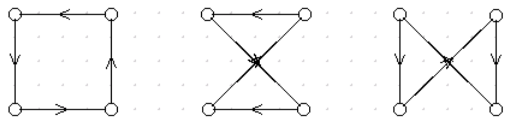
(a) (b)

Fig. 7 Directed graphs for $\boldsymbol{\mathcal{P}}_3$ and $\boldsymbol{\mathcal{P}}_3^T$.



(a) (b) (c)

Fig. 8 Directed graphs for $\boldsymbol{\mathcal{P}}_{41}$, $\boldsymbol{\mathcal{P}}_{42}$, and $\boldsymbol{\mathcal{P}}_{43}$.



(a) (b) (c)

Fig. 9 Directed graphs for $\boldsymbol{\mathcal{P}}_{41}^T$, $\boldsymbol{\mathcal{P}}_{42}^T$, and $\boldsymbol{\mathcal{P}}_{43}^T$.

In any case, the nominal system can be given as follows:

$$\bar{\boldsymbol{x}}(t_k) = \boldsymbol{\Phi}(t_k, t_0)\boldsymbol{x}(t_0) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0) \in \mathbb{Z}^n. \quad (8)$$

In this paper, the event-driven function $\boldsymbol{f}$ is simplified as

$$\varepsilon_{ii}(t_k) = \frac{f_i(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k))}{x_i(t_k)} \in \mathbb{R}. \quad (9)$$

As for a matrix expression, the following can be written:

$$\boldsymbol{\mathcal{E}}(t_k) = \mathrm{diag}\{\varepsilon_{11}(t_k) \cdots \varepsilon_{nn}(t_k)\} \quad (10)$$

On the basis of the above premises, (7) can be written as:

$$\boldsymbol{x}(t_k) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0) + \sum_{l=1}^{u} k \boldsymbol{\mathcal{P}}^{k-l} \boldsymbol{\mathcal{E}}(t_{l-1}) \boldsymbol{x}(t_{l-1}). \quad (11)$$

Here, consider a new type of transition matrix, i.e.,

$$\boldsymbol{\Psi}(k, l) := \boldsymbol{\mathcal{P}}^{k-l} \boldsymbol{\mathcal{E}}(t_{l-1}).u \quad (12)$$

Thus, (11) can be rewritten as follows:

$$\boldsymbol{x}(t_k) = \bar{\boldsymbol{x}}(t_k) + \sum_{l=1}^{k} \boldsymbol{\Psi}(k, l) \boldsymbol{x}(t_{l-1}), \quad (13)$$

where $\bar{\boldsymbol{x}}(t_k) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0)$ is the nominal state response.

5. MULTIPLE METRICS AND INEQUALITIES

The metric in the state space (i.e., vector space) is usually defined by a scalar value. However, it may lead to a severe condition for the stability of some kind of nonlinear systems. Therefore, we consider the metric (i.e., norm) for each element of the state as follows:

$$\|x_i(t_k)\|_{\ell_\infty} := \sup_{1\leq l\leq k} |x_i(t_l)| \in \mathbb{Z}_+. \tag{14}$$

When considering multiple metrics, the following vector can be written:

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} = \begin{bmatrix} \|x_1(t_k)\|_{\ell_\infty} \\ \|x_2(t_k)\|_{\ell_\infty} \\ \vdots \\ \|x_n(t_k)\|_{\ell_\infty} \end{bmatrix} \in \mathbb{Z}_+^n. \tag{15}$$

Based on these considerations, the following inequalities are obtained from (13)[5]:

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty} + \left\| \sum_{l=1}^{k} \boldsymbol{\Psi}(k,l)\boldsymbol{x}(t_{l-1}) \right\|_{\ell_\infty}. \tag{16}$$

Here, we define a matrix expression with positive elements,

$$\|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} := \begin{bmatrix} \|\theta_{11}(t_k)\|_{\ell_\infty} & \cdots & \|\theta_{1n}(t_k)\|_{\ell_\infty} \\ \vdots & \ddots & \vdots \\ \|\theta_{n1}(t_k)\|_{\ell_\infty} & \cdots & \|\theta_{nn}(t_k)\|_{\ell_\infty} \end{bmatrix},$$

where

$$\|\theta_{ij}(t_k)\|_{\ell_\infty} = \left\| \sum_{l=1}^{k} \psi_{ij}(k,l)x_j(t_{l-1}) \right\|_{\ell_\infty} / \|x_j(t_k)\|_{\ell_\infty}$$
$$\|\theta_{ij}(t_k)\|_{\ell_\infty} \in \mathbb{R}_+ \quad i,j = 1,2,\cdots,n.$$

Therefore, inequality (16) can be written as:

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty} + \|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} \cdot \|x(t_k)\|_{\ell_\infty}. \tag{17}$$

Moreover, it can be written as follows:

$$\left(\boldsymbol{I} - \left\|\boldsymbol{\Theta}(t_k)\right\|_{\ell_\infty}\right)\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty}. \tag{18}$$

Here, we consider the following simultaneous "equality":

$$(\boldsymbol{I} - \boldsymbol{C})\boldsymbol{X} = \boldsymbol{Y}, \tag{19}$$

where $\boldsymbol{C} \geqq \boldsymbol{0}$, $\boldsymbol{X} \geqq \boldsymbol{0}$ and $\boldsymbol{Y} \geqq \boldsymbol{0}$ are correspond to $\left\|\boldsymbol{\Theta}(t_k)\right\|_{\ell_\infty}$, $\|\boldsymbol{x}(t_k)\|_{\ell_\infty}$ and $\|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty}$, respectively.

[5]Inequality symbols for matrices and vectors are defined based on [10].

6. STABILITY AND SECURITY CONDITIONS

With respect to the simultaneous equation (19), the following propositions can be obtained [5, 10-14].

Lemma. For any $\boldsymbol{0} \leqq \boldsymbol{Y} < \infty$, we can obtain $\boldsymbol{0} \leqq \boldsymbol{X} < \infty$ if and only if $\boldsymbol{A} = \boldsymbol{I} - \boldsymbol{C}$ is a nonnegative-inverse matrix [10] (i.e., Ostrowski's M-matrix).

Proof. The proof is obtained from the property of M-matrix (i.e., $\boldsymbol{A}^{-1} \geqq \boldsymbol{0}$). □

Based on the above, the stability and security (i.e., boundednes) conditions are given as follows.

Definition. If $\|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty} = \boldsymbol{Y} < \infty$ leads to $\|\boldsymbol{x}(t_k)\|_{\ell_\infty} = \boldsymbol{X} < \infty$ for all $k \in \mathbb{N}$, the event-driven discrete system is defined as stable in a relative sense [5]. Thus, the following theorem is given.

Theorem. If $\boldsymbol{A} = \boldsymbol{I} - \boldsymbol{C} = \boldsymbol{I} - \|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty}$ is a nonnegative-inverse matrix (i.e., Ostrowski's M-matrix), the system is stable (and bounded) in a relative sense. That is, a finite $\boldsymbol{X} \in \mathbb{Z}_+^n$ can be obtained for any $\boldsymbol{Y} \in \mathbb{Z}_+^n$. Thus, the security is of course satisfied.

Proof. The proof is obviouly obtained from the above Lemma. □

Incidentally, the condions of M-matrix for $\boldsymbol{A} = \boldsymbol{I} - \boldsymbol{C}$ are given as follows.

(1) $\rho(\boldsymbol{C}) := \max_{1\leq i\leq n} |\lambda_i| < 1$

(2) The principal minors of $\boldsymbol{A}$ are all positive (i.e., $\Delta_i > 0, \ \ 1 \leq i \leq n$).

Here, λ_i are eigenvalues of $\boldsymbol{C}$.

7. NUMERICAL EXAMPLES

Example 1. Consider the following recurrent third-order system constructed by an irreducible structure matrix [15]:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k + \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}_k \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k$$

The nominal system of this example contains a periodic mode with $p = 3$ as shown in Fig. 7 (a). Here, we assume the event-driven signals e_1, e_2, and e_3 are as shown in Fig. 10. Figure 11 shows state traces from $x_1(0) = -2.0$, $x_2(0) = 2.0$, and $x_3(0) = 1.0$ for $t_k < 200$. The state trace representation in the 3D coordinates is given as shown in Fig. 12. The response is pseudo-periodic and obviously bounded (i.e., relatively stable) for $t_k < 200$.

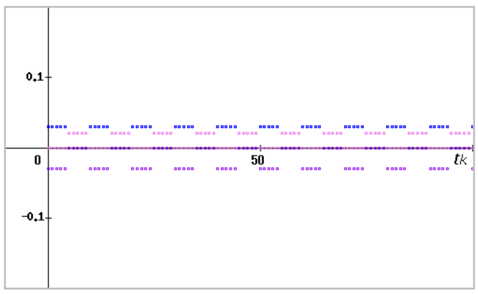


Fig. 10 Time series of event signals, $e_1 = 0.03$, $e_2 = -0.03$, and $e_3 = 0.02$.

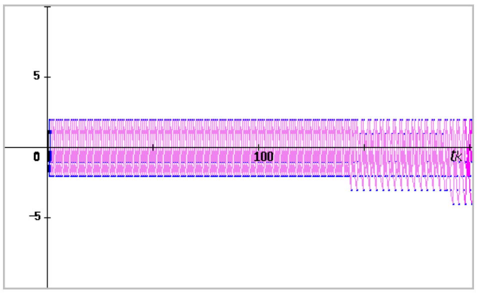


Fig. 11 State traces when $x_1(0) = -2.0$, $x_2(0) = 2.0$, and $x_3(0) = 1.0$.

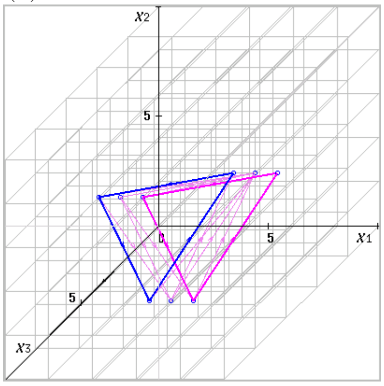


Fig. 12 A state trace in the 3D coordinates when $x_1(0) = -2.0$, $x_2(0) = 2.0$, and $x_3(0) = 1.0$ $(t_k < 200)$.

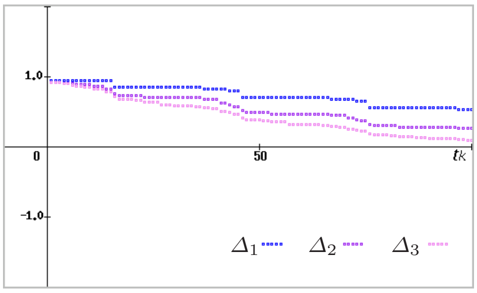


Fig. 13 Time series of $\varDelta_1$, $\varDelta_2$, and $\varDelta_3$.

The stability condition is given below:

$$\begin{cases} \varDelta_1 = 1 - \|\theta_{11}(t_k)\|_{\ell_\infty} > 0 \\ \varDelta_2 = (1 - \|\theta_{11}(t_k)\|_{\ell_\infty})(1 - \|\theta_{22}(t_k)\|_{\ell_\infty}) \\ \quad - \|\theta_{12}(t_k)\|_{\ell_\infty}\|\theta_{21}(t_k)\|_{\ell_\infty} > 0 \\ \varDelta_3 = (1 - \|\theta_{11}(t_k)\|_{\ell_\infty})(1 - \|\theta_{22}(t_k)\|_{\ell_\infty})(1 - \|\theta_{33}(t_k)\|_{\ell_\infty}) \\ \quad - \|\theta_{12}(t_k)\|_{\ell_\infty}\|\theta_{23}(t_k)\|_{\ell_\infty}\|\theta_{31}(t_k)\|_{\ell_\infty} \\ \quad - \|\theta_{13}(t_k)\|_{\ell_\infty}\|\theta_{32}(t_k)\|_{\ell_\infty}\|\theta_{21}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{11}(t_k)\|_{\ell_\infty})\|\theta_{23}(t_k)\|_{\ell_\infty}\|\theta_{32}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{22}(t_k)\|_{\ell_\infty})\|\theta_{13}(t_k)\|_{\ell_\infty}\|\theta_{31}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{33}(t_k)\|_{\ell_\infty})\|\theta_{12}(t_k)\|_{\ell_\infty}\|\theta_{21}(t_k)\|_{\ell_\infty} > 0. \end{cases}$$

Figure 13 shows the time-sequences of $\varDelta_i$ $(i = 1, 2, 3)$. As is obvious from the figure, the stability (bounded) condition will be satisfied (in fact, $\varDelta_i > 0$ for $t_k < 200$).

Next, when considering event signals as shown in Fig. 14, the time series of state traces become as shown in Fig. 15. Fig. 16 shows the state trace in the 3D coordinates. In this case, the boundedness (i.e., securitiy) will not be guaranteed. The time-sequences of $\varDelta_i$ $(i = 1, 2, 3)$ become as shown in Fig. 17.

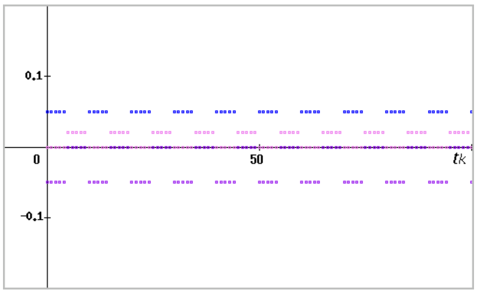


Fig. 14 Time series of event signals, $e_1 = 0.05$, $e_2 = -0.05$, and $e_3 = 0.02$.

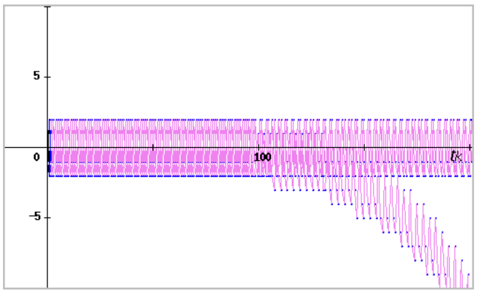


Fig. 15 State traces when $x_1(0) = -2.0$, $x_2(0) = 2.0$, and $x_3(0) = 1.0$.

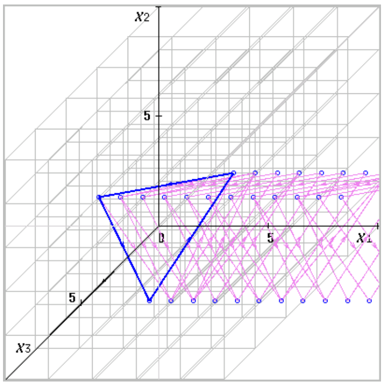


Fig. 16 A state trace in the 3D coordinates when $x_1(0) = -2.0$, $x_2(0) = 2.0$, and $x_3(0) = 1.0$ $(t_k < 200)$.

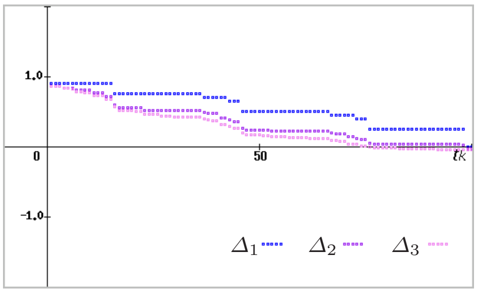


Fig. 17 Time series of $\varDelta_1$, $\varDelta_2$, and $\varDelta_3$.

Example 2. Consider the following fourth-order system with a irreducible structure matrix:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_k + \begin{bmatrix} e_1 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 \\ 0 & 0 & e_3 & 0 \\ 0 & 0 & 0 & e_4 \end{bmatrix}_k \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_k$$

The nominal system of this example contains a periodic mode with $p = 4$ as shown in Fig. 8 (a). In this example, event-driven signals e_1, e_2, and e_3 are assumed

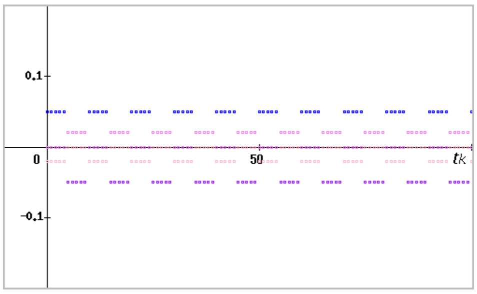


Fig. 18 Time series of event signals, $e_1 = 0.05$, $e_2 = -0.05$, $e_3 = 0.02$, and $e_4 = -0.02$.

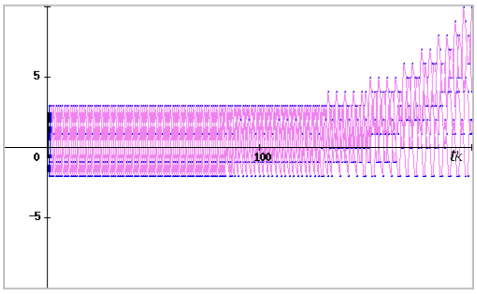


Fig. 19 State traces when $x_1(0) = -3.0$, $x_2(0) = 2.0$, $x_3(0) = 1.0$, and $x_4(0) = -1.0$.

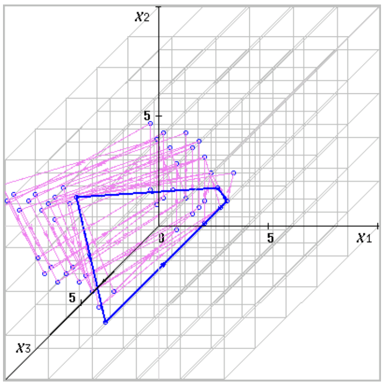


Fig. 20 A state trace in the 3D coordinates when $x_1(0) = -3.0$, $x_2(0) = 2.0$, and $x_3(0) = 1.0$ $x_4(0) = -1.0$ ($t_k < 200$).

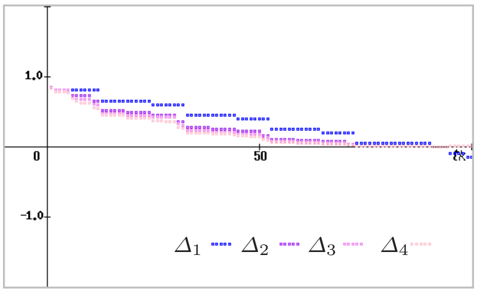


Fig. 21 Time series of Δ_1, Δ_2, Δ_3, and Δ_4.

to be as shown in Fig. 18. Figure 19 shows state traces from $x_1(0) = -3.0$, $x_2(0) = 2.0$, $x_3(0) = 1.0$, and $x_4(0) = -1.0$ for $t_k < 200$. Although the state trace of fourth-order systems cannot be expressed in the 3D coordinates, the general view of variables only x_1, x_2, and x_3 is given in Fig. 20. As are obvious from these figures, the stability and security (boundedness) will be garanteed in a limited range. For a reference, the time-sequences Δ_1, Δ_2, Δ_3, and Δ_4 of this example are shown in Fig. 21..

8. CONCLUSIONS

There are many event-driven types of discrete control systems in practice. Therefore, in this paper, the (finite-time) stability of event-driven control systems has been studied. First, the lattice concept based on ordered sets was reviewed in general. Then, integer lattice coordinates were introduced, and the stability and security (boundedness) of event-driven discrete control systems was analyzed using multiple metrics and simultaneous linear inequalities. As a result, a theorem based on nonnegative marices and M-matrices was derived. Numerical examples clarify the stability and security of event-driven types of discrete control systems. The result will be useful for some discrete event systems.

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