EXERCISES WITH ANSWERS

"Discrete Control Systems"

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Chapter 1 Mathematical Descriptions and Models

(1) Find the homogeneous solutions to each of the following difference equation:

(i) $y(k+2) - 3y(k+1) + 2y(k) = u(k)$,

(ii) $y(k+2) - 2y(k+1) + y(k) = u(k)$.

Ans.: Apply the method of subsection 1.3.4.

(i) Consider the homogeneous equation,

$$y(k+2) - 3y(k+1) + 2y(k) = 0.$$

If $y(k) = \lambda^k$ is assumed as a solution, then

$$(\lambda^2 - 3\lambda + 2)\lambda^k = 0.$$

The characteristic equation is written as

$$\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0.$$

Since the charcteristic roots are $\lambda = 1$ and $\lambda = 2$, the following solution is obtained:

$$y(k) = C_1 + C_2(2)^k, \quad k = 0, 1, 2, \cdots,$$

where C_1 and C_2 are arbitrary constants.

(ii) Consider the homogeneous equation,

$$y(k+2) - 2y(k+1) + y(k) = 0.$$

The characteristic equation is written as

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0.$$

In this case, since the characteristic root is multiple ($\lambda = 1$), the term $k\lambda^k$ should be considered. Thus, the solution is given as

$$y(k) = C_1 + C_{12}k, \quad k = 0, 1, 2, \cdots,$$

where C_1 and C_{12} are arbitrary constants.

(2) Prove that

$$\mathcal{Z}[kf(kh)] = -z\frac{\mathrm{d}F(z)}{\mathrm{d}z} \quad \text{for } |z| > 1.$$

Ans.: From the definition of z-transform (1.30),

$$\mathcal{Z}[kf(kh)] = \sum_{k=0}^{\infty} kf(kh)z^{-k} = f(h)z^{-1} + 2f(2h)z^{-2} + 3f(3h)z^{-3} + \cdots.$$

On the other hand, the following relation is obtained from (1.31) for $|z| > 1$:

$$\frac{\mathrm{d}F(z)}{\mathrm{d}z} = -f(h)z^{-2} - 2f(2h)z^{-3} - 3f(3h)z^{-4} - \cdots.$$

Thus,

$$\mathcal{Z}[kf(kh)] = -z\frac{\mathrm{d}F(z)}{\mathrm{d}z}$$

was proved.

(3) Determine the z-transform of discrete ramp function,

$$f(k) = \begin{cases} k & \text{for } k \geq 0, \\ 0 & \text{for } k < 0. \end{cases}$$

Ans.: From the definition of z-transform (1.30),

$$F(z) = z^{-1} + 2z^{-2} + 3z^{-3} + \cdots.$$

Then,

$$z^{-1}F(z) = z^{-2} + 2z^{-3} + 3z^{-4} + \cdots.$$

Hence, the following closed form is obtained for $|z| > 1$:

$$(1 - z^{-1})F(z) = z^{-1} + z^{-2} + z^{-3} + \cdots = \frac{z^{-1}}{1 - z^{-1}}.$$

Thus,

$$F(z) = \frac{z^{-1}}{(1 - z^{-1})^2} = \frac{z}{(z-1)^2}.$$

This result corresponds to the second line in Table 1.2 when $h = 1$.

(4) Show that an n-th order discrete-time equation (1.10) can also be written as the following vector-matrix form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & \dots & a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

and

$$y(k) = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \dots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k).$$

Here, as a general expression, $b_0 \neq 1$ is considered in the output equation.

Ans.-1: The vector-matrix form given above corresponds to (1.17) and (1.18). The set of first-order equations of those expressions is written as follows:

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = x_3(k) \\ \vdots \\ x_{n-1}(k+1) = x_n(k) \\ x_n(k+1) = -a_n x_1(k) - a_{n-1} x_2(k) - \dots - a_2 x_{n-1}(k) - a_1 x_n(k) + u(k). \end{cases} \tag{11.1}$$

and

$$y(k) = (b_n - a_n b_0) x_1(k) + (b_{n-1} - a_{n-1} b_0) x_2(k) + \dots + (b_1 - a_1 b_0) x_n(k) + b_0 u(k). \tag{11.2}$$

With respect to the set of equations (11.1), we obtain the following expression:

$$x_n(k+1) + a_1 x_n(k) + \dots + a_{n-1} x_2(k) + a_n x_1(k) = u(k), \tag{11.3}$$

that is,

$$x_1(k+n) + a_1 x_1(k+n-1) + \dots + a_{n-1} x_1(k+1) + a_n x_1(k) = u(k).$$

Therefore, in general, when the discrete equation is given by

$$y(k+n) + \dots + a_{n-1} y(k+1) + a_n y(k) = b_0 u(k+n) + \dots + b_{n-1} u(k+1) + b_n u(k)$$

as shown in (1.10), the output $y(k)$ must be written as

$$\begin{aligned} y(k) &= b_0 x_1(k+n) + b_1 x_1(k+n-1) + \dots + b_{n-1} x_1(k+1) + b_n x_1(k) \\ &= b_0 x_n(k+1) + b_1 x_n(k) + \dots + b_{n-1} x_2(k) + b_n x_1(k) \end{aligned}$$

because the above system is linear.

Here, from (11.3)

$$x_n(k+1) = u(k) - a_1 x_n(k) - a_2 x_{n-1}(k) - \cdots - a_n x_1(k).$$

By substituting this expression into (11.4), we can obtain the following output equation (i.e., (11.2):

$$\begin{aligned} y(k) &= b_0[u(k) - a_1 x_n(k) - a_2 x_{n-1} - \cdots - a_n x_1(k)] + b_1 x_n(k) + \cdots + b_n x_1(k) \\ &= (b_n - a_n b_0)x_1(k) + (b_{n-1} - a_{n-1}b_0)x_2(k) + \cdots + (b_1 - a_1 b_0)x_n(k) + b_0 u(k). \end{aligned}$$

Ans.-2: Next, a direct proof is shown. From (11.2), the following equations are given:

$$\begin{cases} y(k) = (b_n - a_n b_0)x_1(k) + \cdots + (b_1 - a_1 b_0)x_n(k) + b_0 u(k) \\ y(k+1) = (b_n - a_n b_0)x_1(k+1) + \cdots + (b_1 - a_1 b_0)x_n(k+1) + b_0 u(k+1) \\ \vdots \\ y(k+n-1) = (b_n - a_n b_0)x_1(k+n-1) + \cdots + (b_1 - a_1 b_0)x_n(k+n-1) + b_0 u(k+n-1) \\ y(k+n) = (b_n - a_n b_0)x_1(k+n) + \cdots + (b_1 - a_1 b_0)x_n(k+n) + b_0 u(k+n) \end{cases}$$

Therefore,

$$\begin{aligned} & y(k+n) + a_1 y(k+n-1) + \cdots + a_{n-1} y(k+1) + a_n y(k) \\ &= (b_n - a_n b_0)[a_n x_1(k) + a_{n-1} x_1(k+1) + \cdots + a_1 x_1(k+n-1) + x_1(k+n)] \\ &\quad + (b_{n-1} - a_{n-1} b_0)[a_n x_1(k) + a_{n-1} x_1(k+1) + \cdots + a_1 x_1(k+n-1) + x_1(k+n)] \\ &\quad \vdots \\ &\quad + (b_1 - a_1 b_0)[a_n x_n(k) + a_{n-1} x_{n-1}(k+1) + \cdots + a_1 x_n(k+n-1) + x_n(k+n)] \\ &\quad + a_n b_0 u(k) + a_{n-1} b_0 u(k+1) + \cdots + a_1 b_0 u(k+n-1) + b_0 u(k+n) \\ &= b_0 u(k+n) + b_1 u(k+n-1) + \cdots + b_{n-1} u(k+1) + b_n u(k). \end{aligned}$$

This completes the proof.

As for simple case $n = 2$, the following results are given.

Ans.-1:

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = -a_2 x_1(k) - a_1 x_2(k) + u(k) \end{cases} \tag{11.4}$$

and

$$y(k) = (b_2 - a_2 b_0)x_1(k) + (b_1 - a_1 b_0)x_2(k) + b_0 u(k). \tag{11.5}$$

From (11.4),

$$x_2(k+1) + a_1 x_2(k) + a_2 x_1(k) = u(k), \tag{11.6}$$

that is,

$$x_1(k+2) + a_1 x_1(k+1) + a_2 x_1(k) = u(k).$$

Therefore, when the discrete equation is given by

$$y(k+2) + a_1 y(k+1) + a_2 y(k) = b_0 u(k+2) + b_1 u(k+1) + b_2 u(k).$$

the output $y(k)$ must be written as

$$\begin{aligned} y(k) &= b_0 x_1(k+2) + b_1 x_1(k+1) + b_2 x_1(k) \\ &= b_0 x_2(k+1) + b_1 x_2(k) + b_2 x_1(k). \end{aligned}$$

Here, from (11.6)

$$x_2(k+1) = u(k) - a_1 x_2(k) - a_2 x_1(k).$$

Thus

$$y(k) = b_0 u(k) + (b_1 - a_1 b_0) x_2(k) + (b_2 - a_2 b_0) x_1(k).$$

Ans.-2 From (11.2),

$$\begin{cases} y(k) = (b_2 - a_2 b_0) x_1(k) + (b_1 - a_1 b_0) x_2(k) + b_0 u(k) \\ y(k+1) = (b_2 - a_2 b_0) x_1(k+1) + (b_1 - a_1 b_0) x_2(k+1) + b_0 u(k+1) \\ y(k+2) = (b_2 - a_2 b_0) x_1(k+2) + (b_1 - a_1 b_0) x_2(k+2) + b_0 u(k+2). \end{cases}$$

Therefore,

$$\begin{aligned} & y(k+2) + a_1 y(k+1) + a_2 y(k) \\ &= (b_2 - a_2 b_0)[a_2 x_1(k) + a_1 x_1(k+1) + x_1(k+2)] \\ &\quad + (b_1 - a_1 b_0)[a_2 x_2(k) + a_2 x_2(k+1) + x_2(k+2)] + a_2 b_0 u(k) + a_1 b_0 u(k+1) + b_0 u(k) \\ &= b_0 u(k+2) + b_1 u(k-1) + b_2 u(k). \end{aligned}$$

This completes the proof for $n = 2$.

(5) Determine the discrete-time version (1.78) of the following state space representation:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{cases} \tag{11.7}$$

Assume $h = 1$, and $u = \mathrm{const.}$ through $0 \le t \le h$.

Ans. Let us calculate the transition matrix $\boldsymbol{\Phi}(\tau)$ from (1.76), i.e.,

$$\begin{aligned} \boldsymbol{\Phi}(\tau) &= \boldsymbol{I} + \boldsymbol{A}\tau + \frac{\boldsymbol{A}^2 \tau^2}{2!} + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \tau + \frac{1}{2!} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^2 \tau^2 + \frac{1}{3!} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^3 \tau^3 + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \tau + \frac{1}{2!} \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix} \tau^2 + \frac{1}{3!} \begin{bmatrix} 6 & 7 \\ -14 & -15 \end{bmatrix} \tau^3 + \cdots \end{aligned}$$

Then, each element of the matrix is written as follows:

$$\begin{cases} \Phi_{11}(\tau) = 1 - \tau^2 + \tau^3 + \cdots \\ \Phi_{12}(\tau) = \tau - \frac{1}{2}\tau^3 + \frac{7}{6}\tau^3 + \cdots \\ \Phi_{21}(\tau) = -2\tau + 3\tau^2 - \frac{7}{3}\tau^3 + \\ \Phi_{22}(\tau) = 1 - 3\tau + \frac{7}{2}\tau^2 - \frac{5}{2}\tau^3 + \cdots . \end{cases} \tag{11.8}$$

On the other hand, in regard to Laplace transform for (11.7), i.e.,

$$\begin{bmatrix} \hat{x}_1(s) \\ \hat{x}_2(s) \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}(s) = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{(s+2)(s+1)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}(s)$$

the following transition matrix is obtained:

$$\Phi(t) = \mathcal{L}^{-1}\{(s\boldsymbol{I} - \boldsymbol{A})^{-1}\} = \begin{bmatrix} -\mathrm{e}^{-2t} + 2\mathrm{e}^{-t} & -\mathrm{e}^{-2t} + \mathrm{e}^{-t} \\ 2\mathrm{e}^{-2t} - 2\mathrm{e}^{-t} & 2\mathrm{e}^{-2t} - \mathrm{e}^{-t} \end{bmatrix}.$$

Each element of this matrix is equivalent to (11.8) for $t = \tau$ in the closed form.

In (1.78),

$$\boldsymbol{\Gamma}(h) = \int_0^h \boldsymbol{\Phi}(\tau)\boldsymbol{B}\mathrm{d}\tau = \int_0^h \begin{bmatrix} -\mathrm{e}^{-2\tau} + \mathrm{e}^{-\tau} \\ 2\mathrm{e}^{-2\tau} - \mathrm{e}^{-\tau} \end{bmatrix} \mathrm{d}\tau = \begin{bmatrix} \mathrm{e}^{-2\tau}/2 - \mathrm{e}^{-\tau} \\ -\mathrm{e}^{-2\tau} + \mathrm{e}^{-\tau} \end{bmatrix}_0^h.$$

Since $h = 1$,

$$\boldsymbol{\Gamma(h)} = \begin{bmatrix} (\mathrm{e}^{-2} - 1)/2 - (\mathrm{e}^{-1} - 1) \\ (-\mathrm{e}^{-2} + 1) + (\mathrm{e}^{-1} - 1) \end{bmatrix}.$$

Thus,

$$\begin{cases} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -\mathrm{e}^{-2} + 2\mathrm{e}^{-1} & -\mathrm{e}^{-2} + \mathrm{e}^{-1} \\ 2\mathrm{e}^{-2} - 2\mathrm{e}^{-1} & 2\mathrm{e}^{-2} - \mathrm{e}^{-1} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} (\mathrm{e}^{-2} - 1)/2 - (\mathrm{e}^{-1} - 1) \\ (-\mathrm{e}^{-2} + 1) + (\mathrm{e}^{-1} - 1) \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix}. \end{cases}$$

$$\begin{cases} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.60 & 0.23 \\ -1.0 & -0.64 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.20 \\ 0.23 \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix}. \end{cases}$$

In this expression, with respect to the input sequence ($u(0)$, $u(1)$, $u(2)$, $\cdots$), the output sequence ($y(0) = x(1)$, $y(1) = x_1(2)$, $y(2) = x_1(3)$, $\cdots$) is determined.

However, if we define the output equation as

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix},$$

With respect to the input sequence $(u(0), u(1), u(2), \cdots)$, the output sequence $(y(0) = x(0),\ y(1) = x_1(1),\ y(2) = x_1(2),\ \cdots)$ is determined. That is, a one-step delay response is obtained as shown in Fig. 1.5.

(6) Determine the z-transform of the following time-function as a closed form:

(i) $f(t) = t \cdot \mathrm{e}^{-at}$

(ii) $f(t) = \sin \omega t$

Ans.(i)
From the definition of z-transform,

$$\begin{aligned} F(z) &= \sum_{k=0}^{\infty} kh\mathrm{e}^{-akh} z^{-k} = h\mathrm{e}^{-ah} z^{-1} + 2h\mathrm{e}^{-2ah} z^{-2} + 3h\mathrm{e}^{-3ah} z^{-3} + \cdots \\ &= h\mathrm{e}^{-ah} z^{-1}(1 + 2\mathrm{e}^{-ah} z^{-1} + 3\mathrm{e}^{-2ah} z^{-2} + \cdots) \end{aligned} \tag{11.9}$$

Multiplying the both sides of (11.9) by $\mathrm{e}^{-ah} z^{-1}$, we obtain

$$\mathrm{e}^{-ah} z^{-1} F(z) = h\mathrm{e}^{-ah} z^{-1}(\mathrm{e}^{-ah} z^{-1} + 2\mathrm{e}^{-2ah} z^{-2} + \cdots) \tag{11.10}$$

Subtracting (11.10) from (11.9),

$$(1 - \mathrm{e}^{-ah} z^{-1}) F(z) = h\mathrm{e}^{-ah} z^{-1}(1 + \mathrm{e}^{-ah} z^{-1} + \mathrm{e}^{-2ah} z^{-2} + \cdots).$$

Thus, we can obtain the following closed form when $|\mathrm{e}^{ah} z| > 1$:

$$F(z) = \frac{h\mathrm{e}^{-ah} z^{-1}}{(1 - \mathrm{e}^{-ah} z^{-1})^2}. = \frac{h\mathrm{e}^{-ah} z}{(z - \mathrm{e}^{-ah})^2}.$$

This result corresponds to the fouth line in Table 1.2.

Ans. (ii)
From the definition of z-transform,

$$\begin{aligned} F(z) &= \sum_{k=0}^{\infty} (\sin k\omega h) z^{-k} = \sum_{k=0}^{\infty} \left(\frac{\mathrm{e}^{jk\omega h} - \mathrm{e}^{-jk\omega h}}{2j} \right) z^{-k}, \qquad j = \sqrt{-1} \\ &= \frac{1}{2j} \left[(1 + \mathrm{e}^{j\omega h} z^{-1} + \mathrm{e}^{2j\omega h} z^{-2} + \cdots) - (1 + \mathrm{e}^{-jk\omega h} z^{-1} + \mathrm{e}^{-2j\omega h} z^{-2} + \cdots) \right] \\ &= \frac{1}{2j} \left(\frac{1}{1 - \mathrm{e}^{j\omega h} z^{-1}} - \frac{1}{1 - \mathrm{e}^{-j\omega h} z^{-1}} \right) \\ &= \frac{(\sin \omega h) z^{-1}}{1 - 2(\cos \omega h) z^{-1} + z^{-2}} = \frac{(\sin \omega) z}{z^2 - 2(\cos \omega h) z + 1}. \end{aligned}$$

Of course, the convergent condition, $|z| > 1$, must be satisfied. This result corresponds to the fifth line in Table 1.2.

(7) Determine $f(k)$ for

$$F(z) = \frac{z+1}{z^3 - 2z^2 + 1.5z - 0.5},$$

and confirm that it corresponds to the delayed response as shown in light blue in Fig. 1.5.

Ans. By applying the result of direct division, (1.51),

$$c_k = b_k - \sum_{i=0}^{k-1} a_{k-i} c_i,$$

the following coefficients are obtained.

$$\begin{aligned}
c_0 &= b_0 = 0 \\
c_1 &= b_1 = 0 \\
c_2 &= b_2 = 1 \\
c_3 &= b_3 - a_1 c_2 = 1 + 2 = 3 \\
c_4 &= -a_2 c_2 - a_1 c_3 = -1.5 + 6 = 4.5 \\
&\dots
\end{aligned}$$

This time sequence corresponds to the (light blue) response as shown in Fig. 1.5.

Chapter 2 Discretized Feedback Systems

(1) Prove that the sector condition in (2.5),

$$|g(e)| \leq \beta |e|$$

is equivalently written as (2.7), i.e.,

$$-\beta e^2 \leq g(e)e \leq \beta e^2. \tag{12.1}$$

Ans.:
Multiplying the both sides of (2.5) by $|e|$,

$$|g(e)e| \leq \beta e^2.$$

Then,

$$-\beta e^2 \leq g(e)e \leq \beta e^2.$$

(2) Confirm that block diagram Fig. 2.16 is equivalent to Fig. 2.15.

Ans.: From Fig. 2.15,

$$\begin{cases} \hat{e}^*(z) = (1 + q\delta(z))\hat{e}(z) \\ \hat{w}^*(z) = \hat{w}(z) - \beta q \delta(z) \hat{e}(z). \end{cases}$$

These equations are based on (2.20), (2.21), and (2.22), i.e.,

$$\frac{(1+z^{-1})}{2}\hat{e}^*(z) = \frac{(1+z^{-1})}{2}\hat{e}(z) + q\cdot\frac{(1-z^{-1})}{h}\hat{e}(z),$$
$$\frac{(1+z^{-1})}{2}\hat{w}^*(z) = \frac{(1+z^{-1})}{2}\hat{w}(z) - \beta q\cdot\frac{(1-z^{-1})}{h}\hat{e}(z).$$

$$\delta(z) = \frac{2}{h}\cdot\frac{1-z^{-1}}{1+z^{-1}}.$$

Thus, from the following equations,

$$\begin{cases} \hat{e}^*(z) = (1+q\delta(z))\hat{e}(z) \\ \hat{w}(z) = \hat{w}^*(z) + \beta q\delta(z)\hat{e}(z), \end{cases}$$

Fig. 1.16 can easily be obtained.

(3) From Fig. 2.18, determine the loop transfer function $H(\beta, q, z)$ in Fig. 2.19.

Ans.: In Fig. 2.18, the following equations hold:

$$\begin{cases} \hat{w}(z) = \hat{w}^*(z) + \beta q\delta(z)\hat{e}(z) \\ \hat{v}(z) = \hat{w}(z) + K\hat{e}(z) \\ \hat{e}(z) = \hat{r}(z) - G(z)\hat{v}(z) - G(z)\hat{d}(z) \end{cases} .$$

Therefore, (2.24)-(2.26) are obtained.

(4) From (2.5) and (2.29), prove that the sector inequality in (2.31), that is,

$$0 \le \frac{f(e(k))}{e(k)} \le 2\beta. \qquad (12.2)$$

Ans.: From (2.5), (2.29), and (12.1), i.e.,

$$|g(e)| \le \beta|e|,$$

$$f(e) = g(e) + \beta e,$$

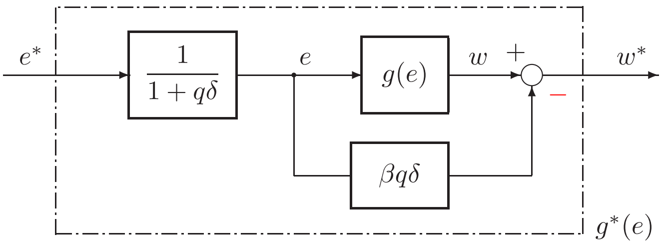

Fig. 2.15 Transformation of nonlinear element $g(e)$.

and

$$-\beta e^2 \leq g(e)e \leq \beta e^2, \tag{12.3}$$

From (12.3),

$$0 \leq (g(e) + \beta e)e = f(e)e \leq 2\beta e^2.$$

Thus (12.2) can easily obtained for $e \neq 0$. This completes the proof.

(5) Prove Lemma 2.1, that is,

$$\|\overline{w}^\dagger(k)\|_{2,N} \leq \beta\|\overline{e}^\dagger(k)\|_{2,N} \leq \beta\|\overline{e}(k)\|_{2,N}$$

using inequality (2.34).

Ans.: From (2.34), i.e.,

$$\begin{cases} |\overline{w}^\dagger(1)|^2 \leq \beta^2|\overline{e}^\dagger(1)|^2 \leq \beta|\overline{e}(1)|^2 \\ |\overline{w}^\dagger(2)|^2 \leq \beta^2|\overline{e}^\dagger(2)|^2 \leq \beta|\overline{e}(2)|^2 \\ \vdots \\ |\overline{w}^\dagger(N)|^2 \leq \beta^2|\overline{e}^\dagger(N)|^2 \leq \beta|\overline{e}(N)|^2 \end{cases}$$

Therefore,

$$\left(\sum_{k=1}^{N} |\overline{w}^\dagger(k)|^2\right)^{1/2} \leq \beta \left(\sum_{k=1}^{N} |\overline{e}^\dagger(k)|^2\right)^{1/2} \leq \beta \left(\sum_{k=1}^{N} |\overline{e}(k)|^2\right)^{1/2}.$$

This completes the proof.

(6) For $N = 2$, prove Schwarz's inequality (2.60).

Ans.: In regard to $N = 2$, the following is obtained:

$$\left(\sum_{k=1}^{2} |x(k)|^2\right)\left(\sum_{k=1}^{2} |y(k)|^2\right) - \left(\sum_{k-1}^{2} |x(k)y(k)|\right)^2$$

$$|x(1)|^2|y(1)|^2 + |x(2)|^2|y(2)|^2 + |x(1)|^2|y(2)|^2 + |x(2)|^2|y(1)|^2$$
$$- |x(1)y(1)|^2 - |x(2)y(2)|^2 - 2|x(1)y(1)| \cdot |x(2)y(2)|$$

$$= |x(1)y(2) - x(2)y(1)|^2 \geq 0.$$

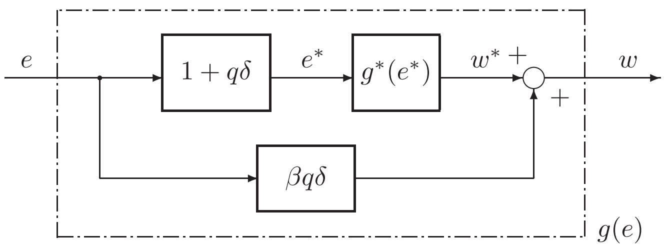

Fig. 2.16 Equivalent nonlinear subsystem.

This completes the proof.

(7) Using the result of (6), prove Minkowski's inequality (2.68) when $N = 2$.

Ans.: In regard to sequences $x(k)$, $v(k)$ $(k = 1, 2)$ and $y(k)$, $v(k)$ $(k = 1, 2)$, Schwartz's inequalities for $N = 2$ are given as

$$\begin{aligned}(|x(1)v(1)| + |x(2)v(2)|)^2 &\leq (|x(1)|^2 + |x(2)|^2)(|v(1)|^2 + |v(2)|^2)\\ (|y(1)v(1)| + |y(2)v(2)|)^2 &\leq (|y(1)|^2 + |y(2)|^2)(|v(1)|^2 + |v(2)|^2).\end{aligned}$$

Taking the square root of both sides of this inequality,

$$\begin{aligned}|x(1)||v(1)| + |x(2)||v(2)| &\leq \sqrt{(|x(1)|^2 + |x(2)|^2)(|v(1)|^2 + |v(2)|^2)}\\ |y(1)||v(1)| + |y(2)||v(2)| &\leq \sqrt{(|y(1)|^2 + |y(2)|^2)(|v(1)|^2 + |v(2)|^2)}.\end{aligned}$$

By adding these inequalities, we have

$$\begin{aligned}&(|x(1)| + |y(1)|)|v(1)| + (|x(2)| + |y(2)|)|v(2)|\\ &\quad \leq \left(\sqrt{|x(1)|^2 + |x(2)|^2} + \sqrt{|y(1)|^2 + |y(2)|^2}\right)\left(\sqrt{|v(1)|^2 + |v(2)|^2}\right).\end{aligned}$$

If $v(k)$ $(k = 1, 2)$ are written as

$$v(k) = x(k) + y(k),$$

the above inequalities are given as

$$|v(1)|^2 + |v(2)|^2 \leq \left(\sqrt{|x(1)|^2 + |x(2)|^2} + \sqrt{|y(1)|^2 + |y(2)|^2}\right)\left(\sqrt{|v(1)|^2 + |v(2)|^2}\right).$$

By deviding the both sides by $\sqrt{|v(1)|^2 + |v(2)|^2}$, we obtain the following Minkowski's inequality for $N = 2$:

$$\sqrt{(|x(1) + y(1)|)^2 + (|x(2) + y(2)|)^2} \leq \sqrt{|x(1)|^2 + |x(2)|^2} + \sqrt{|y(1)|^2 + |y(2)|^2}.$$

Chapter 3 Robust Stability Analysis

(1) Show that inequality (3.15) is corresponding to Fig. 3.2 (b).

Ans.: If we define $U(\omega) = \Re\{1/G(\mathrm{e}^{j\omega h}\}$ and $V(\omega) = \Im\{1/G(\mathrm{e}^{j\omega h}\}$, the following inequality is obtained:

$$(U(\omega) + K)^2 + V^2(\omega) > \rho^2,$$

where ρ is the radius of the small circle in Fig. 3.2 (b). The plotting, $(U(\omega), V(\omega))$, in the figure is called *inverse Nyquist locus*.

(2) Using the result of (3.39), prove the following inequality:

$$\beta < K \cdot \frac{-q\Omega V + \sqrt{q^2\Omega^2 V^2 + (U^2+V^2)\{(1+U)^2+V^2\}}}{U^2+V^2}. \tag{13.1}$$

Ans.: From (3.39),

$$\beta^2(1+q^2\Omega^2)(U^2+V^2) < (K+KU-\beta q\Omega V)^2 + (KV+\beta q\Omega U)^2$$

$$\beta^2(U^2+V^2) < K^2(1+U)^2 - 2K\beta q\Omega V(1+U) + K^2V^2 + 2\beta Kq\Omega UV.$$

Therefore,

$$\beta^2(U^2+V^2) + 2\beta Kq\Omega V - K^2\{(1+U)^2+V^2\} < 0.$$

Since $\beta \geq 0$,

$$\beta < \frac{-Kq\Omega V + \sqrt{K^2q^2\Omega^2V^2 + K^2(U^2+V^2)\{(1+U)^+V^2\}}}{U^2+V^2}.$$

This completes the proof.

(3) Show that the right side of the above inequality can be written as

$$K \cdot \frac{-q\Omega\sin\theta + \sqrt{q^2\Omega^2\sin^2\theta + \rho^2 + 2\rho\cos\theta + 1}}{\rho}, \tag{13.2}$$

where $\rho(\omega) = |KG(\mathrm{e}^{j\omega h})|$ and $\theta(\omega) = \angle KG(\mathrm{e}^{j\omega h})$.

Ans.: Obviously,

$$U(\omega) = \rho\cos\theta, \qquad V(\omega) = \rho\sin\theta.$$

The right side of (13.1) is written as

$$K \cdot \frac{-q\Omega\rho\sin\theta + \sqrt{q^2\Omega^2\rho^2\sin^2\theta + \rho^2\{(1+\rho\cos\theta)^2 + \rho^2\sin^2\theta\}}}{\rho^2}.$$

This result is equivalent to (13.2).

(4) Prove the stability condition (3.11) in Theorem 3.1 based on the definition of input-output stability.

Ans.: From (3.1), i.e.,

$$\begin{aligned} &v(k) = f(e,k), \quad k = 0,1,2,\cdots \\ &\hat{y}(z) = G(z)\hat{u}(z) \\ &e(k) = r(k) - y(k), \quad u(k) = v(k) + d(k), \end{aligned} \tag{13.3}$$

the following expressions are obtained by using Minkowski's inequality:

$$|v(k)| = |f(e,k)| \leq \rho|e(k)|, \qquad \|v(k)\|_2 \leq \rho\|e(k)\|_2.$$

$$|\hat{y}(z)| = |G(z)||\hat{y}(z)| \le \sup_{|z|=1} |G(z)||\hat{u}(z)|, \;\; (z = \mathrm{e}^{j\omega h}, \; -\pi/h < \omega < \pi/h), \tag{13.4}$$

and

$$\|e(k)\|_2 \le \|r(k)\|_2 + \|y(k)\|_2, \quad \|u(k)\|_2 \le \|v(k)\|_2 + \|d(k)\|_2,$$

where $\|\cdot\|_2$ is a norm in the ℓ_2 space.

$$\|\cdot\|_2 = \left(\sum_{k=0}^{\infty} |\cdot|^2\right)^{1/2}.$$

Therefore,

$$\|u(k)\|_2 \le \rho \|e(k)\|_2 + \|d(k)\|_2.$$

Here, on the basis of Parseval's identity, the following relations are obtained:

$$\|\hat{y}(z)\|_2 = \|y(k)\|_2, \quad \|\hat{u}(z)\|_2 = \|u(k)\|_2, \qquad \text{for } \; z = \mathrm{e}^{j\omega h}.$$

Therefore, from (13.3) and (13.4), we have

$$\begin{aligned}\|e(k)\|_2 &\le \|r(k)\|_2 + \sup_{|z|=1} |G(z)| \cdot \|u(k)\|_2 \\ &\le \|r(k)\|_2 + \sup_{|z|=1} |G(z)|(\rho \cdot \|e(k)\|_2 + \|d(k)\|_2)\end{aligned}$$

and

$$(1 - \rho \cdot \sup_{|z|=1} |G(z)|)\|e(k)\|_2 \le \|r(k)\|_2 + \sup_{|z|=1} |G(z)| \cdot \|d(k)\|_2.$$

Thus, if $\|r(k)\|_2 < \infty$ and $\|d(k)\|_2 < \infty$, then $\|e(k)\|_2 < \infty$ when

$$\sup_{|z|=1} |G(z)| < \frac{1}{\rho}.$$

Here, it is assumed that

$$|G(z)| < \infty \quad \text{for } \; z = \mathrm{e}^{j\omega h}.$$

(5) Prove that

$$\delta(\mathrm{e}^{j\omega h}) = j\Omega(\omega) = j\frac{2}{h}\tan\left(\frac{\omega h}{2}\right)$$

in (3.33), where

$$\delta(z) = \frac{2}{h} \cdot \frac{z-1}{z+1}.$$

Ans.: When $z = \mathrm{e}^{j\omega h}$,

$$\begin{aligned}\delta(\mathrm{e}^{j\omega h}) &= \frac{2}{h} \cdot \frac{\mathrm{e}^{j\omega h} - 1}{\mathrm{e}^{j\omega h} + 1} = \frac{2}{h} \cdot \frac{\cos\omega h + j\sin\omega h - 1}{\cos\omega h + j\sin\omega h + 1} \\ &= \frac{2}{h} \cdot \frac{(\cos\omega h - 1 + j\sin\omega h)(\cos\omega h + 1 - j\sin\omega h)}{(\cos\omega h + 1)^2 + \sin^2\omega h} \\ &= \frac{2}{h} \cdot \frac{j\sin\omega h}{\cos\omega h + 1} = \frac{2}{h} \cdot \frac{2j\sin(\omega h/2)\cos(\omega h/2)}{2\cos^2(\omega h/2) - 1 + 1} = j\frac{2}{h} \cdot \tan\left(\frac{\omega h}{2}\right).\end{aligned}$$

This completes the proof.

(6) Show that (3.89) is equivalent to (3.47).

Ans.: This question may be natural, because $\xi(q,\omega)$ is the inverse function of $\eta(q,\omega)$. Here, calculate (3.89) in regard to (3.90) directly.

$$\begin{aligned}\frac{\partial\xi}{\partial q} &= \frac{\rho}{(-q\Omega\sin\theta+\sqrt{q^2\Omega^2\sin^2\theta+\rho^2+2\rho\cos\theta+1})^2}\\ &\cdot\left(-\Omega\sin\theta+\frac{q\Omega^2\sin^2\theta}{\sqrt{q^2\Omega^2\sin^2\theta+\rho^2+2\rho\cos\theta+1}}\right)\\ &= \frac{-\rho}{(-q\Omega\sin\theta+\sqrt{q^2\Omega^2\sin^2\theta+\rho^2+2\rho\cos\theta+1})^2}\\ &\cdot\left(1-\frac{q\Omega\sin\theta}{\sqrt{q^2\Omega^2\sin^2\theta+\rho^2+2\rho\cos\theta+1}}\right)\Omega\sin\theta\\ &= \frac{-\xi(q,\omega)\Omega(\omega)\sin\theta}{\sqrt{q^2\Omega^2\sin^2\theta+\rho^2+2\rho\cos\theta+1}}\end{aligned}$$

Thus, since $V=\rho\sin\theta$, it can be seen that

$$\left(\frac{\partial\xi(q,\omega)}{\partial q}\right)_{q=q_0,\omega=\omega_0}=0$$

is equivalent to (3.47) because $\Omega(\omega_0)>0$ and $\xi((q_0,\omega_0)>0$ for $0<\omega_0<\pi/h$.

(7) Derive (3.116) and (3.118) from (3.114) and (3.115).

Ans.: This question is a classical problem for the Hall diagram. Squaring the both sides of (3.114),

$$M^2=\frac{U^2+V^2}{(1+U)^2+V^2}.$$

Then,

$$(M^2-1)U^2+2UM^2+M^2+(M^2-1)V^2=0)$$

Thus, if $M\neq 1$,

$$\left(U+\frac{M^2}{M^2-1}\right)^2+V^2=\frac{M^4}{(M^2-1)^2}-\frac{M^2}{M^2-1}=\left(\frac{M}{M^2-1}\right)^2.$$

This completes the proof.

Chapter 4 Model Reference Feedback and PID Control

(1) Confirm that the following equality holds in regard to (4.3) and (4.6):

$$\lim_{h\to 0}C(\delta)=C(s).$$

Ans.: This is a simple limit problem. Since $z = \mathrm{e}^{hs}$,

$$\delta = \frac{2}{h} \cdot \frac{1 - (1 - hs + \cdots)}{1 + (1 - hs + \cdots)}.$$

Then,

$$\lim_{h \to 0} \delta = s, \qquad \lim_{h \to 0} \delta^{-1} = s^{-1}.$$

Therefore, $\lim\limits_{h \to 0} C(\delta)$ becomes $C(s)$.

(2) Determine the z-transform of the following plant with a zero-order hold:

$$G(s) = \frac{K}{(s+1)(s+2)}, \quad K = 1.0$$

Assume the sampling period $h = 0.1$, and use the result of Example 4.1 (A), i.e., Eqs. (4.15)-(4.17).

Ans.: As was shown in (4.16), including the integration of the zero-order hold, the following partial fraction is obtained:

$$G_1(s) = \frac{G(s)}{s} = \frac{0.5}{s} - \frac{1}{s+1} + \frac{0.5}{s+2}.$$

Then,

$$G_1(z) = \frac{0.5}{1 - z^{-1}} - \frac{1}{1 - \mathrm{e}^{-1} z^{-1}} + \frac{0.5}{1 - \mathrm{e}^{-2} z^{-1}}.$$

The z-transform of $(*4.1)$ with the zero-order hold is given as

$$G(z) = (1 - z^{-1}) G_1(z) = \frac{0.0045z + 0.0041}{z^2 - 1.724z + 0.7408}.$$

(3) Determine the characteristic equation of the nominal system, $\mathcal{F}(z) = 0$, for Example 4.1 (A) when a PI controller is used (i.e., case (i), $K_p = 1.0$, $C_I = 1.0$, and $C_D = 0.0$).

Ans.: Since PI controller is written as

$$C(z) = 1 + \frac{1}{1 - z^{-1}} = \frac{2z - 1}{z - 1},$$

the characteristic equation is given by

$$\begin{aligned} \mathcal{F}(z) &= (z-1)(z^2 - 1.489z; 0.549) + (2z-1)(0.0082z + 0.0067) \\ &= z^3 - 2.47z^2 + 2.04z - 0.56 = 0. \end{aligned}$$

(4) Show that the approximate PID control system in Fig. 4.18 is obtained from the model-reference feedback system in Fig. 4.17, when $\mathcal{D}_m(\cdot)$ and $\mathcal{D}_f(\cdot)$, are in high resolution.

Ans.: When the discretized elements are in high resolution, $\mathcal{D}_m(\cdot)$ and $\mathcal{D}_f(\cdot)$ are considered identity functions. Therefore, the following equations can be obtained:

$$\hat{u}_3(z) = \frac{K_m}{1 + C_1\delta + C_2\delta^2} \cdot \hat{u}_2(z) + \hat{v}_1^\dagger(z) + \hat{d}'(z)$$
$$\hat{u}_2(z) = \frac{1 + C_1\delta + C_2\delta^2}{K_m(1 + c_1\delta + c_2\delta^2)} \cdot \hat{u}_3(z) + \hat{r}'(z).$$

Then,

$$\hat{u}_2(z) = \frac{1}{1 + c_1\delta + c_2\delta^2} \cdot \hat{u}_2(z) + \frac{1 + C_1\delta + C_2\delta^2}{K_m(1 + c_1\delta + c_2\delta^2)} (\hat{v}_1^\dagger(z) + \hat{d}'(z)) + \hat{r}'(z),$$

and

$$\hat{u}_2(z) = \frac{1 + C_1\delta + C_2\delta^2}{K_m(c_1\delta + c_2\delta^2)} (\hat{v}_1^\dagger(z) + d'(z)) + \frac{1 + c_1\delta + c_2\delta^2}{c_1\delta + c_2\delta^2} \cdot \hat{r}'(z).$$

Thus, $C(\delta)$ and $D(\delta)$ can be defined as shown in Fig. 4.18. Here, d' is considered a discretization error.

(5) Regarding the simultaneous (linear) inequalities,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le y_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \le y_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \le y_3, \end{cases}$$

show that the following operations are valid (a similar concept to (1), (2), and (3) in Appendix A of Chap. 1):

(1) interchaging two inequalities,

(2) multiplying each term in one inequality by a positive constant,

(3) adding a positive multiple of one inequality to another.

Ans.:

(1) An example of simple operation is:

$$\begin{cases} a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \le y_2 \\ a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le y_1 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \le y_3. \end{cases} \tag{14.1}$$

Up to here, the reader will think that there is no problem. However, if variables were replaced as $x_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow y_2$, i.e.,

$$\begin{cases} a_{22}x_1 + a_{21}x_2 + a_{23}x_3 \le y_1 \\ a_{12}x_1 + a_{11}x_2 + a_{13}x_3 \le y_2 \\ a_{32}x_1 + a_{31}x_2 + a_{33}x_3 \le y_3, \end{cases}$$

the reader may be a little confused by this expression. In the following questions, it should be noted that the numbering of variables in a mathematical model is not so important.

(2) With respect to (15.2), consider the following simple inequalities:

$$f(x) \leq g(y), \quad \text{that is,} \quad g(y) - f(x) \geq 0.$$

When multiplying both sides of them by a positive constant ($k > 0$), the following inequalities are naturally valid:

$$k(g(y) - f(x)) \geq 0, \quad \text{that is,} \quad kf(x) \leq kg(y).$$

Then, the proposition was proved.

(3) The equivalency for a positive multiple of one inequality was described in (2). Therefore, the problem of adding one inequality to another is considered here. In general, with respect to the following two inequalities:

$$\begin{cases} f_1(x) \leq g_1(y), & \text{i.e.,} \quad g_1(y) - f_1(x) \geq 0, \\ f_2(x) \leq g_2(y), & \text{i.e.,} \quad g_2(y) - f_2(x) \geq 0, \end{cases}$$

the following inequality will be satisfied:

$$g_1(y) + g_2(y) - f_1(x) - f_2(x) \geq 0, \quad \text{i.e.,} \quad f_1(x) + f_2(x) \leq g_1(y) + g_2(y).$$

Here, an example of a set of linear equations is shown below. Consider the following two inequalities:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \leq y_1 \tag{14.2}$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \leq y_2 \quad . \tag{14.3}$$

If $a_{11} > 0$ and $a_{21} \leq 0$ (i.e., $-a_{21}/a_{11} \geq 0$), the first inequality (ref*5.1) can be given as follows:

$$-a_{21}x_1 - \frac{a_{21}a_{12}}{a_{11}}x_2 - \frac{a_{21}a_{13}}{a_{11}}x_3 \leq -\frac{a_{21}}{a_{11}}y_1. \tag{14.4}$$

By adding (14.4) to (14.3), the following inequality is obtained:

$$\left(a_{22} - \frac{a_{21}a_{12}}{a_{11}}\right)x_2 + \left(a_{23} - \frac{a_{21}a_{13}}{a_{11}}\right)x_3 \leq y_2 - \frac{a_{21}}{a_{11}}y_1. \tag{14.5}$$

That is, a variable (x_1) has been eliminated. When multiplying both sides of (14.5) by a_{11}, the following inequality is obtained:

$$(a_{11}a_{22} - a_{12}a_{21})x_2 + (a_{11}a_{23} - a_{13}a_{21})x_3 \leq a_{11}y_2 - a_{21}y_1.$$

(6) In regard to a vector-matrix expression, $\boldsymbol{Ax} \leqq \boldsymbol{y}$, where

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

show that the following operations are valid (a similar concept to (a), (b), and (c)) in Appendix A of Chap. 1):

(a) interchanging two rows,

(b) multiplying each term in one row by a positive constant,

(c) adding a positive multiple of one row to another,

considering the influence on variables x_i and y_j $(i, j = 1, 2, 3)$.

Ans.:

(a) An example of interchaging two row is given as

$$\boldsymbol{A}' = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{y}' = \begin{bmatrix} y_2 \\ y_1 \\ y_3 \end{bmatrix}.$$

When interchanging two variables (i.e., $x_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow y_2$), the above matrix can be expressed as

$$\boldsymbol{A}^* = \begin{bmatrix} a_{22} & a_{21} & a_{23} \\ a_{12} & a_{11} & a_{13} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}, \quad \boldsymbol{x}^* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{y}^* = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

(b) The following exapression will be satisfied for $k > 0$ from (5).

$$\boldsymbol{A}^\dagger = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{y}^\dagger = \begin{bmatrix} ky_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

(c) From the comments in (5), if $a_{11} > 0$, $a_{21} \leq 0$, and $a_{31} \leq 0$ (i.e., $-a_{21}/a_{11} \geq 0$ and $-a_{31}/a_{11} \geq 0$), the following inequality for a matrix expression is obtained:

$$\boldsymbol{A}^{(2)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{y}^{(2)} = \begin{bmatrix} y_1 \\ y_2^{(2)} \\ y_3^{(2)} \end{bmatrix},$$

where

$$a_{22}^{(2)} = \left(a_{22} - \frac{a_{21}a_{12}}{a_{11}}\right), \quad a_{23}^{(2)} = \left(a_{23} - \frac{a_{21}a_{13}}{a_{11}}\right)$$
$$a_{32}^{(2)} = \left(a_{32} - \frac{a_{31}a_{12}}{a_{11}}\right), \quad a_{33}^{(2)} = \left(a_{33} - \frac{a_{31}a_{13}}{a_{11}}\right)$$
$$y_2^{(2)} = y_2^{(1)} - (a_{21}/a_{11})y_1, \quad y_3^{(2)} = y_3 - (a_{31}/a_{11})y_1.$$

Furthermore, If $a_{22}^{(2)} > 0$ and $a_{32}^{(2)} < 0$ (i.e., $-a_{32}^{(2)}/a_{22}^{(2)} > 0$), the following inequality for vector-matrix expression can be obtained:

$$\boldsymbol{A}^{(3)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & 0 & a_{33}^{(3)} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{y}^{(3)} = \begin{bmatrix} y_1 \\ y_2^{(2)} \\ y_3^{(3)} \end{bmatrix},$$

where

$$a_{33}^{(3)} = \left(a_{33}^{(2)} - \frac{a_{32}^{(2)} a_{23}^{(2)}}{a_{22}^{(2)}} \right)$$
$$y_3^{(3)} = y_3^{(2)} - (a_{32}^{(2)}/a_{22}^{(2)}) y_3^{(2)}.$$

The expression of $\boldsymbol{A}^{(3)}$ is equivalent to (4.79), and is referred to as an upper triangular matrix.

(7) Show that the condition of M-matrix is not influenced by the above operations (a), (b), and (c).

Ans.: From the proof of of Theorem 4.1, the condiltion of M-matrix for $\boldsymbol{A}$ is given below.

$$a_{11}^{(1)} = \Delta_1 = a_{11} > 0$$

$$a_{22}^{(2)} = \frac{\Delta_2}{\Delta_1} = \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}{a_{11}} > 0$$

$$a_{33}^{(3)} = \frac{\Delta_3}{\Delta_2} = \frac{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} > 0, \qquad \text{where } \ a_{ij} \leq 0 \ (i \neq j).$$

(a) With repect to interchanged matrix A^*, the above condition will be written as follows:

$$a_{11}^{(1)} = \Delta_1^* = a_{22} > 0$$

$$a_{22}^{(2)} = \frac{\Delta_2^*}{\Delta_1^*} = \frac{\begin{vmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{vmatrix}}{a_{22}} > 0$$

$$a_{33}^{(3)} = \frac{\Delta_3^*}{\Delta_2^*} = \frac{\begin{vmatrix} a_{22} & a_{21} & a_{23} \\ a_{12} & a_{11} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} > 0.$$

Here, $a_{22} > 0$ must be satisfied, because $a_{11} > 0$ and $a_{11}a_{22} - a_{12}a_{21} > 0$. Moreover, $\varDelta_2^* = \varDelta_2$ and $\varDelta_3^* = \varDelta_3$. Thus, it can be seen that the condition of M-matrix is not influenced by interchanging two rows.

(b) When multiplying each element in one row (e.g. 1st row) of $\boldsymbol{A}$ by $k > 0$, principal minors of matrix $\boldsymbol{A}$ are given as

$$\varDelta_1^\dagger = k\varDelta_1, \quad \varDelta_2^\dagger = \begin{vmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{vmatrix} = k\varDelta_2, \quad \varDelta_3^\dagger = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k\varDelta_3.$$

Therefore, the condition of M-matrix will not be influenced by multiplying a positive constant.

(c) From the postscript of (6) that was developed based on the operation (c), the following results for principal minors of matrix $\boldsymbol{A}^{(3)}$ are obtained:

$$\varDelta_1 = a_{11}, \quad \varDelta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \varDelta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

That is, it can be seen that the condition of M-matrix is not be influenced by the operation (c).

Chapter 5 Multi-Loop Feedback Systems

(1) From Fig. 5.4, determine the 2×2 loop transfer matrix which corresponds to $\boldsymbol{W}(\boldsymbol{\beta}, \boldsymbol{q}, z)$ in Fig. 5.8.

Ans.: From (5.44), $\boldsymbol{W}(\boldsymbol{\beta}, \boldsymbol{q}, z)$ is written as follows:

$$(\boldsymbol{I} + \boldsymbol{q}\delta)[\boldsymbol{I} + (\boldsymbol{K} + \boldsymbol{\beta}\boldsymbol{q}\delta)\boldsymbol{G}(z)\boldsymbol{C}(z)]^{-1}\boldsymbol{G}(z)\boldsymbol{C}(z),$$

where

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{q} = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix},$$

$$\boldsymbol{G}(z) = \begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{bmatrix}, \quad \boldsymbol{C}(z) = \begin{bmatrix} C_1(z) & 0 \\ 0 & C_2(z) \end{bmatrix}.$$

Since

$$\boldsymbol{G}(z)\boldsymbol{C}(z) = \begin{bmatrix} G_{11}(z)C_1(z) & G_{12}C_2(z) \\ G_{21}(z)C_1(z) & G_{22}C_2(z) \end{bmatrix}$$

$$\begin{aligned} &\boldsymbol{I} + (\boldsymbol{K} + \boldsymbol{\beta}\boldsymbol{q}\delta)\boldsymbol{G}(z)\boldsymbol{C}(z) \\ &= \begin{bmatrix} 1 + (K_1 + \beta_1 q_1 \delta)G_{11}(z)C_1(z) & (K_1 + \beta_1 q_1 \delta)G_{12}(z)C_2(z) \\ (K_2 + \beta_2 q_2 \delta)G_{21}(z)C_1(z) & 1 + (K_2 + \beta_2 q_2 \delta)G_{22}(z)C_2(z) \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned}
&[\boldsymbol{I} + (\boldsymbol{K} + \boldsymbol{\beta q}\delta)\boldsymbol{G}(z)\boldsymbol{C}(z)]^{-1} \\
&= \begin{bmatrix} 1 + (K_2 + \beta_2 q_2 \delta)G_{22}(z)C_2(z) & -(K_1 + \beta_1 q_1 \delta)G_{12}(z)C_2(z) \\ -(K_2 + \beta_2 q_2 \delta)G_{21}(z)C_1(z) & 1 + (K_1 + \beta_1 q_1 \delta)G_{11}(z)C_1(z) \end{bmatrix} / \Delta.
\end{aligned}$$

Here,

$$\begin{aligned}
\Delta = {} & [1 + (K_1 + \beta_1 q_1 \delta)G_{11}(z)C_1(z)] \cdot [1 + (K_2 + \beta_2 q_2 \delta)G_{22}(z)C_2(z)] \\
& - (K_1 + \beta_1 q_1 \delta)(K_2 + \beta_2 q_2 \delta)G_{12}(z)G_{21}(z)C_1(z)C_2(z).
\end{aligned}$$

Moreover,

$$\boldsymbol{I} + \boldsymbol{q}\delta = \begin{bmatrix} 1 + q_1\delta & 0 \\ 0 & 1 + q_2\delta \end{bmatrix}.$$

(2) Show that if all principal minors in (5.15) and (5.48) are positive, all diagonal elements become positive.

Ans.: The condition of M-matrix must not be influenced by interchanging variables, i.e., $x_i \leftrightarrow x_j$ $(i \neq j)$. Thus, a_{kk} $(k = 2, \cdots, n)$ will become positive when $a_{11} > 0$. Nevertheless, the above proposition will be proved as follows. Obviously, the following expression is valid from (5.17):

$$\begin{cases}
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{n-1,n} \end{bmatrix} x_n = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_{n-1} \end{bmatrix} \\
\begin{bmatrix} a_{n1} & a_{n2} & \cdots & a_{n,n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} + a_{nn} x_n = \tilde{y}_n.
\end{cases} \tag{15.1}$$

Therefore, if all principal minors of matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{bmatrix}$$

in (15.1) are positive (i.e., the $(n-1) \times (n-1)$ matrix is M-matrix), $0 \leq x_i < \infty$ $(i = 1, 2, \cdots, n-1)$ can be obtained for $a_{ij} \leq 0$ $(i \neq j)$ and $0 \leq x_n < \infty$. Thus, $a_{nn} > 0$ must be satisfied for $\tilde{y}_n \geq 0$.

(3) Prove that if the solution of the left side of simultaneous equations (5.16) is calculated using some numerical method as $0 \leq x_1 < \infty$, $0 \leq x_2 < \infty$, $\cdots$, $0 \leq x_n < \infty$ (in regard to $\tilde{y}_j \geq 0$, $j = 1, 2, \cdots, n$ and $a_{ij} \leq 0$, $i \neq j$), then the matrix (5.15) becomes an M-matrix.

Ans.: From (5.17),

$$\begin{cases} a_{11}^{(1)} x_1 + a_{12}^{(1)} x_2 + \cdots + a_{1n}^{(1)} x_n = \tilde{y}_1^{(1)} \\ a_{22}^{(2)} x_2 + \cdots + a_{2n}^{(2)} x_n = \tilde{y}_2^{(2)} \\ \vdots \\ a_{nn}^{(n)} x_n = \tilde{y}_n^{(n)}. \end{cases}$$

Then,

$$\begin{cases} a_{11}^{(1)} x_1 = \tilde{y}_1^{(1)} - a_{12}^{(1)} x_2 - \cdots - a_{1n}^{(1)} x_n \\ a_{22}^{(2)} x_2 = \tilde{y}_2^{(2)} - a23^{(2)} x_3 - \cdots + a_{2n}^{(2)} x_n = \\ \vdots \\ a_{nn}^{(n)} x_n = \tilde{y}_n^{(n)}. \end{cases}$$

Thus, the following conditions can be obtained:

$$a_{11}^{(1)} = a_{11} > 0, \quad a_{22}^{(2)} > 0, \quad \cdots \quad a_{nn}^{(n)} > 0$$

as shown in (5.18).

(4) The colored area shown in Fig. 5.26 is given by the following (vector-matrix) inequality:

$$\begin{bmatrix} 0.8 & -0.3 & -0.3 \\ -0.1 & 0.8 & -0.2 \\ -0.1 & -0.3 & 0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leqq \begin{bmatrix} 3.6 \\ 3.9 \\ 2.6 \end{bmatrix} \tag{15.2}$$

Confirm that the matrix of the left side of the above inequality is an M-matrix, and determine that $0 \leq x_1 < \infty$, $0 \leq x_2 < \infty$, and $0 \leq x_3 < \infty$.

Ans.: Based on (5.17) and (5.18) (i.e., (4.79) in Chapter 4), $a_{11}^{(1)}$, $a_{22}^{(2)}$, and $a_{33}^{(3)}$ for (15.2) are as follows:

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & 0 & a_{33}^{(3)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \tilde{y}_1^{(1)} \\ \tilde{y}_2^{(2)} \\ \tilde{y}_3^{(3)} \end{bmatrix}. \tag{15.3}$$

Here,

$$a_{11}^{(1)} = 0.8, \quad a_{12}^{(1)} = -0.3, \quad a_{13}^{(1)} = -0.3$$

$$\begin{cases} a_{22}^{(2)} = \dfrac{1}{0.8} \begin{vmatrix} 0.8 & -0.3 \\ -0.1 & 0.8 \end{vmatrix} = 0.7625, \quad a_{23}^{(2)} = \dfrac{1}{0.8} \begin{vmatrix} 0.8 & -0.3 \\ -0.1 & -0.2 \end{vmatrix} = -0.2375, \\ a_{32}^{(2)} = \dfrac{1}{0.8} \begin{vmatrix} 0.8 & -0.3 \\ -0.1 & -0.3 \end{vmatrix} = -0.3375, \quad a_{33}^{(2)} = \dfrac{1}{0.8} \begin{vmatrix} 0.8 & -0.3 \\ -0.1 & 0.6 \end{vmatrix} = 0.5625, \end{cases}$$

and

$$a_{33}^{(3)} = \frac{1}{0.7625}\begin{vmatrix} 0.7625 & -0.2375 \\ -0.3375 & 0.5625 \end{vmatrix} = 0.4574,$$

Then, the right side of (15.3) can be written as

$$\begin{bmatrix} 0.8 & -0.3 & -0.3 \\ 0 & 0.7625 & -0.2375 \\ 0 & 0 & 0.4574 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \tilde{y}_1^{(1)} \\ \tilde{y}_2^{(2)} \\ \tilde{y}_3^{(3)} \end{bmatrix}.$$

Obviously,

$$\begin{cases} a_{11}^{(1)} = 0.8 > 0 \\ a_{22}^{(2)} = \dfrac{\begin{vmatrix} 0.8 & -0.3 \\ -0.1 & 0.8 \end{vmatrix}}{0.8} = \dfrac{0.61}{0.8} = 0.7625 > 0 \\ a_{33}^{(3)} = \dfrac{\begin{vmatrix} 0.8 & -0.3 & -0.3 \\ -0.1 & 0.8 & -0.2 \\ -0.1 & -0.3 & 0.6 \end{vmatrix}}{\begin{vmatrix} 0.8 & -0.3 \\ -0.1 & 0.8 \end{vmatrix}} = \dfrac{0.279}{0.61} = 0.4574 > 0. \end{cases}$$

Therefore, the matrix is an M-matrix.

Moreover,

$$\begin{cases} \tilde{y}_1^{(1)} = 3.6 \\ \tilde{y}_2^{(2)} = 3.9 - \dfrac{-0.1}{0.8} \cdot 3.6 = 4.35 \\ \tilde{y}_3^{(3)} = 2.6 - \dfrac{-0.1}{0.8} \cdot 3.6 - \dfrac{-0.3375}{0.7625} \cdot 4.35 = 4.9754. \end{cases}$$

Thus, the following solutions are obtained in reverse order:

$$x_3 = 10.878, \quad x_2 = 9.093, \quad x_1 = 11.989.$$

(5) Replace x_2 with x_3 ($x_2 \leftrightarrow x_3$) and write the vector-matrix inequality that corresponds to the above inequality in (4). Confirm that the matrix of the left side of this inequality is an M-matrix.

Ans.: The vector-matrix expression of the replaced inequalities is given as

$$\begin{bmatrix} 0.8 & -0.3 & -0.3 \\ -0.1 & 0.6 & -0.3 \\ -0.1 & -0.2 & 0.8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leqq \begin{bmatrix} 3.6 \\ 2.6 \\ 3.9 \end{bmatrix}.$$

Then,

$$\begin{cases} a_{11}^{(1)} = 0.8 > 0 \\ a_{22}^{(2)} = \dfrac{\begin{vmatrix} 0.8 & -0.3 \\ -0.1 & 0.6 \end{vmatrix}}{0.8} = \dfrac{0.45}{0.8} = 0.5625 > 0 \\ a_{33}^{(3)} = \dfrac{\begin{vmatrix} 0.8 & -0.3 & -0.3 \\ -0.1 & 0.6 & -0.3 \\ -0.1 & -0.2 & 0.8 \end{vmatrix}}{\begin{vmatrix} 0.8 & -0.3 \\ -0.1 & 0.6 \end{vmatrix}} = \dfrac{0.279}{0.45} = 0.62 > 0. \end{cases}$$

It can be seen that the matrix is an M-matrix.

(6) For a general vector-matrix expression, $\boldsymbol{Ax} \leqq \boldsymbol{y}$, where

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

show that the following operations are valid in general:

(a) interchanging two rows,

(b) multiplying each term in one row by a positive constant,

(c) adding a positive multiple of one row to another,

considering the influence on variables x_i and y_j $(i, j = 1, 2, \cdots, n)$.

Ans.

(a) An example of interchanging two row is given as

$$\boldsymbol{A}' = \begin{bmatrix} a_{21} & a_{22} & \dots & a_{2n} \\ a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_2 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

When interchanging two variables (i.e., $x_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow y_2$), the above matrix can be written as

$$\boldsymbol{A}^* = \begin{bmatrix} a_{22} & a_{21} & \dots & a_{2n} \\ a_{11} & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

(b) The following expression will be valid for $k > 0$:

$$\boldsymbol{A}^{\dagger} = \begin{bmatrix} ka_{22} & ka_{21} & \dots & ka_{2n} \\ a_{11} & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} ky_1 \\ ky_2 \\ \vdots \\ ky_n \end{bmatrix}.$$

(c) As shown in Chapter 4 (and (5.17)), if $a_{ij} > 0$ for $i = j$ and $a_{ij} \leq 0$ for $i \neq j$, the following processes will be valid:

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \dots & a_{2n}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}^{(n)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \leqq \begin{bmatrix} y_1^{(1)} \\ y_2^{(2)} \\ \vdots \\ y_n^{(n)} \end{bmatrix}, \tag{15.4}$$

where

$$\begin{cases} a_{ij}^{(2)} = \dfrac{1}{a_{11}^{(1)}} \begin{vmatrix} a_{11}^{(1)} & a_{1j}^{(1)} \\ a_{i1}^{(1)} & a_{ij}^{(1)} \end{vmatrix} \\ a_{ij}^{(3)} = \dfrac{1}{a_{22}^{(2)}} \begin{vmatrix} a_{22}^{(2)} & a_{2j}^{(2)} \\ a_{i2}^{(2)} & a_{ij}^{(2)} \end{vmatrix} \\ \vdots \\ a_{ij}^{(n)} = \dfrac{1}{a_{n-1,n-1}^{(n-1)}} \begin{vmatrix} a_{n-1,n-1}^{(n-1)} & a_{n-1,j}^{(n-1)} \\ a_{i,n-1}^{(n-1)} & a_{ij}^{(n-1)} \end{vmatrix} \quad (i, j = 2, 3, \cdots, n)). \end{cases}$$

Moreover, the following are valid for the right side of inequality (15.4):

$$\begin{cases} y_1^{(1)} = y_1 \\ y_2^{(2)} = y_2^{(1)} - \dfrac{a_{21}^{(1)}}{a_{11}^{(1)}} y_1^{(1)} \\ y_3^{(3)} = y_3^{(1)} - \dfrac{a_{31}^{(1)}}{a_{11}^{(1)}} y_1^{(1)} - \dfrac{a_{32}^{(2)}}{a_{22}^{(2)}} y_2^{(2)} \\ \vdots \\ y_n^{(n)} = y_n^{(1)} - \dfrac{a_{n1}^{(1)}}{a_{11}^{(1)}} y_1^{(1)} - \dfrac{a_{n2}^{(2)}}{a_{22}^{(2)}} y_2^{(2)} - \cdots - \dfrac{a_{n,n-1}^{(n-1)}}{a_{n-1,n-1}^{(n-1)}} y_{n-1}^{(n-1)}. \end{cases}$$

(7) Show that the condition of the M-matrix is not influenced by the above operations (a), (b), and (c) in general. (Refer to Exercises (5) and (6) in Chapter 4.)

Ans. Basic facts about determinants are given as follows (e.g., J. M. Ortega, Matrix Theory - A Second Course -, Plenum Press, 1987, p.12):

(i) If any two rows or columns of $\boldsymbol{A}$ are interchanged, the sign of the determinant changes, but the magnitude remains unchanged.

(ii) If any row or column of $\boldsymbol{A}$ is multiplied by a scalar k, then the determinant is multiplied by k.

(iii) if a scalar multiple of row (or column) of $\boldsymbol{A}$ is added to another row (or column), the determinant remains unchanged.

From the above facts, it can be easily shown that the condition of the M-matrix (i.e., $a_{11}^{(1)} > 0$, $a_{22}^{(2)} > 0$), $a_{33}^{(3)} > 0$, $\cdots$, $a_{nn}^{(n)} > 0$ in (5.18)) is not influenced by the operations (a), (b), and (c).

Chapter 6 Interval Polynomials and Robust Performance

(1) Show the relationship between characteristic equation $\tilde{F}(z) = 0$ and $\tilde{F}(\delta) = 0$, where

$$\tilde{F}(z) = z^3 - [1.3, 1.2]z^2 + [0.7, 0.8]z - [0.2, 0.1] \tag{16.1}$$

and

$$\delta = \frac{2}{h} \cdot \frac{z-1}{z+1}.$$

Ans. First, let us define the following variable:

$$\delta^* = \frac{h}{2}\delta.$$

Then, (16.1), i.e.,

$$\tilde{F}(z) = z^3 - [1.3, 1.2]z^2 + [0.7, 0.8]z - [0.2, 0.1] = 0$$

can be expressed as

$$\tilde{F}(\delta^*) = \left(\frac{1+\delta^*}{1-\delta^*}\right)^3 - [1.3, 1.2]\left(\frac{1+\delta^*}{1-\delta^*}\right)^2 + [0.7, 0.8]\left(\frac{1+\delta^*}{1-\delta^*}\right) - [0.2, 0.1] = 0. \tag{16.2}$$

Multiplying both sides of (16.2) by $(1-\delta^*)^3$, the following interval polynomial (and equation) is obtained:

$$\begin{aligned}\tilde{F}(\delta) := (1-\delta^*)^3\tilde{F}(\delta^*) = (1+\delta^*)^3 - [1.3, 1.2](1+\delta^*)^2(1-\delta^*)\\ + [0.7, 0.8](1+\delta^*)(1-\delta^*)^2 - [0.2, 0.1](1-\delta^*)^3 = 0,\end{aligned}$$

which is a similar expression to (6.26) and (6.27).

(2) Show the relationship between pseudo-sectorial loci in the z-plane and sectorial lines in the s-plane.

Ans. Sectorial and pseudo-sectorial areas in the s- and z-planes are given as shown in Fig. 6.3. Here, pseudo-sectorial loci are given from (6.34) as

$$z = \frac{1+(\gamma+j)\theta}{1-(\gamma+j)\theta} = 1 + 2(\gamma+j)\theta + 2(\gamma+j)^2\theta^2 + 2(\gamma+j)^3\theta^3 + \cdots, \tag{16.3}$$

where $\theta = \omega h/2$. On the other hand, sectorial lines in the s-plane are written as

$$z = \mathrm{e}^{(\gamma+j)\omega h} = \mathrm{e}^{2(\gamma+j)\theta} = 1+2(\gamma+j)\theta+2(\gamma+j)^2\theta^2+\frac{4}{3}(\gamma+j)^3\theta^3+\cdots. \tag{16.4}$$

Obviously, (16.3) is approximately equal to (16.4) for $\theta \to 0$ (i.e., $\omega \to 0$ or $h \to 0$).

(3) With respect to the 2×2 feedback system as shown in Fig. 5.4, determine the (interval) characteristic equation when $N_{d1}(e_1)/e_1$ and $N_{d2}(e_2)/e_2$ are replaced with interval gains $[K_1^-, K_1^+]$ and $[K_2^-, K_2^+]$, respectively.

Ans. According to the expression of Fig. 6.1 and Example 6.1, the characteristic equation of the interval system is given as (6.15), i.e.,

$$\begin{aligned} 1 &+ [K_1^-, K_1^+]C_1(z)G_{11}(z) + [K_2^-, K_2^+]C_2(z)G_{22}(z) \\ &+ [K_1^-, K_1^+] \cdot [K_2^-, K_2^+]C_1(z)C_2(z)G_{11}(z)G_{22}(z) \\ &- [K_1^-, K_1^+] \cdot [K_2^-, K_2^+]C_1(z)C_2(z)G_{12}(z)G_{21}(z). \end{aligned}$$

If each transfer function in Fig. 5.4 is expressed by the numerator and denominator polynomials such as

$$\begin{cases} G_{11}(z) = \dfrac{N_{11}(z)}{D_{11}(z)}, \quad G_{12}(z) = \dfrac{N_{12}(z)}{D_{12}(z)}, \quad G_{21}(z) = \dfrac{N_{21}(z)}{D_{21}(z)}, \quad G_{22}(z) = \dfrac{N_{22}(z)}{D_{22}(z)}, \\ C_1(z) = \dfrac{N_{c1}(z)}{D_{c1}(z)}, \quad C_1(z) = \dfrac{N_{c2}(z)}{D_{c2}(z)}, \end{cases}$$

the characteristic equation is written as

$$\begin{aligned} \tilde{F}(z) &= D_{c1}(z)D_{c2}(z)D_{11}(z)D_{22}(z)D_{12}(z)D_{21}(z) \\ &+[K_1^-, K_1^+]N_{c1}(z)N_{11}(z)D_{c2}(z)D_{22}(z)D_{12}(z)D_{21}(z) \\ &+[K_2^-, K_2^+]N_{c2}(z)N_{22}(z)D_{c1}(z)D_{11}(z)D_{12}(z)D_{21}(z) \\ &+[K^-, K^+]N_{c1}(z)N_{c2}(z)N_{11}(z)N_{22}(z)D_{12}(z)D_{21}(z) \\ &-[K^-, K^+]N_{c1}(z)N_{c2}(z)N_{12}(z)N_{21}(z)D_{11}(z)D_{22}(z) = 0, \end{aligned}$$

where

$$[K^-, K^+] = [K_1^- K_2^-, K_1^+ K_2^+] = [K_1^-, K_1^+] \cdot [K_2^-, K_2^+].$$

(4) Show that the characteristic equations (6.6) and (6.7) are equivalent when $\boldsymbol{C}(z)$ and $\tilde{\boldsymbol{K}} \in [\boldsymbol{K}^-, \boldsymbol{K}^+]$ are diagonal matrix with nonzero diagonal elements.

Ans. Multiplying the square matrix in (6.6),

$$\boldsymbol{I} + \boldsymbol{G}(z)\boldsymbol{C}(z)[\boldsymbol{K}^-, \boldsymbol{K}^+], \tag{16.5}$$

by

$$\boldsymbol{\mathcal{F}} = [\boldsymbol{K}^-, \boldsymbol{K}^+]\boldsymbol{C}(z), \quad (0 < |\det\{\boldsymbol{\mathcal{F}}\}| < \infty)$$

from the left and by

$$\boldsymbol{\mathcal{F}}^{-1} = [\boldsymbol{K}^-, \boldsymbol{K}^+]^{-1}\boldsymbol{C}^{-1}(z), \quad (0 < |\det\{\boldsymbol{\mathcal{F}}^{-1}\}| < \infty)$$

from the right. Then, (16.5) becomes

$$\boldsymbol{I} + [\boldsymbol{K}^-, \boldsymbol{K}^+]\boldsymbol{C}(z)\boldsymbol{G}(z).$$

Thus the following expression is obtained:

$$\det\{\boldsymbol{I} + [\boldsymbol{K}^-, \boldsymbol{K}^+]\boldsymbol{C}(z)\boldsymbol{G}(z)\} = 0.$$

(5) Prove Lemma 6.2 using (6.53).

Ans. From section 6.6,

$$\begin{cases} f_0^{(i)}(\upsilon) = a_{0,0}^{(i)}\upsilon^n + \cdots + a_{0,n-1}^{(i)}\upsilon + a_{0,n}^{(i)} \\ f_1^{(i)}(\upsilon) = b_{0,0}^{(i)}\upsilon^n + \cdots + b_{0,n-1}^{(i)}\upsilon + b_{0,n}^{(i)} \\ f_2^{(i)}(\upsilon) = a_{1,1}^{(i)}\upsilon^{n-1} + \cdots + a_{1,n-1}^{(i)}\upsilon + a_{1,n}^{(i)} \\ f_3^{(i)}(\upsilon) = b_{1,1}^{(i)}\upsilon^{n-1} + \cdots + b_{1,n-1}^{(i)}\upsilon + b_{1,n}^{(i)} \\ \quad \cdots \\ f_{2n}^{(i)} = a_{n,n}. \end{cases}$$

Therefore, the sequence of ratios (6.53),

$$\begin{cases} \lim\limits_{\upsilon\to+\infty} \dfrac{f_1^{(i)}(\upsilon)}{|\upsilon| f_2^{(i)}(\upsilon)} = \lim\limits_{\upsilon\to+\infty} \dfrac{b_{0,0}^{(i)} + b_{0,1}^{(i)}\upsilon^{-1} + \cdots}{|\upsilon|(a_{1,1}^{(i)}\upsilon^{-1} + a_{1,2}^{(i)}\upsilon^{-2} + \cdots)} = \dfrac{b_{0,0}^{(i)}}{a_{1,1}^{(i)}} \\ \lim\limits_{\upsilon\to+\infty} \dfrac{f_3^{(i)}(\upsilon)}{|\upsilon| f_4^{(i)}(\upsilon)} = \lim\limits_{\upsilon\to+\infty} \dfrac{b_{1,1}^{(i)} + b_{1,2}^{(i)}\upsilon^{-1} + \cdots}{|\upsilon|(a_{2,2}^{(i)}\upsilon^{-1} + a_{2,3}^{(i)}\upsilon^{-2} + \cdots)} = \dfrac{b_{1,1}^{(i)}}{a_{2,2}^{(i)}} \\ \quad \cdots \\ \lim\limits_{\upsilon\to+\infty} \dfrac{f_{2n-1}^{(i)}(\upsilon)}{|\upsilon| f_{2n}^{(i)}(\upsilon)} = \lim\limits_{\upsilon\to+\infty} \dfrac{b_{n-1,n-1}^{(i)} + b_{n-1,n}^{(i)}\upsilon^{-1}}{|\upsilon| a_{n,n}^{(i)}\upsilon^{-1}} = \dfrac{b_{n-1,n-1}^{(i)}}{a_{n,n}^{(i)}} \end{cases}.$$

Thus, the sequense shown in (6.54) is obtained. □

(6) Show that if and only if four (corner) polynomials,

$$\begin{cases} F^{(1)}(s) = a_0^+ s^3 + a_1^+ s^2 + a_2^- s + a_3^- \\ F^{(2)}(s) = a_0^- s^3 + a_1^+ s^2 + a_2^+ s + a_3^- \\ F^{(3)}(s) = a_0^+ s^3 + a_1^- s^2 + a_2^- s + a_3^+ \\ F^{(4)}(s) = a_0^- s^3 + a_1^- s^2 + a_2^+ s + a_3^+ \end{cases} \tag{16.6}$$

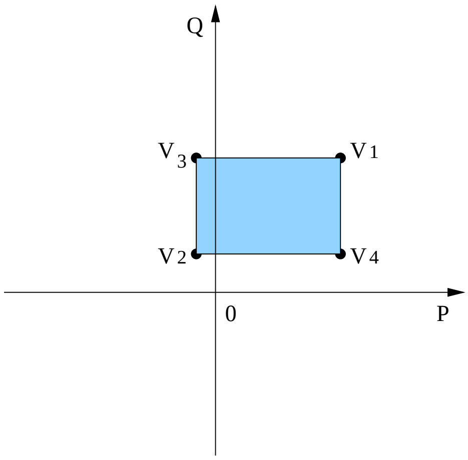

Fig. 16.1 A Kharitonov rectangle.

are stable, the following interval polynomial

$$\tilde{F}(s) = [a_0^-, a_0^+]s^3 + [a_1^-, a_1^+]s^2 + [a_2^-, a_2^+]s + [a_3^-, a_3^+] \tag{16.7}$$

is stable.

Ans. This question corresponds to an example of the problem of Kharitonov Rectangle. (See Appendix 6A.) When considering the imaginary axis, $s = j\omega$ ($\omega > 0$), four corner polynomials (16.6) are written as

$$\begin{cases} F^{(1)}(j\omega) = P^-(\omega) + jQ^-(\omega) \\ F^{(2)}(j\omega) = P^-(\omega) + jQ^+(\omega) \\ F^{(3)}(j\omega) = P^+(\omega) + jQ^-(\omega) \\ F^{(4)}(j\omega) = P^+(\omega) + jQ^+(\omega), \end{cases}$$

where

$$P^-(\omega) = a_3^- - a_1^+\omega^2, \qquad P^+(\omega) = a_3^+ - a_1^-\omega^2,$$
$$Q^-(\omega) = a_2^-\omega - a_0^+\omega^3, \qquad Q^+(\omega) = a_2^+\omega - a_0^-\omega^3.$$

On the other hand, interval polynomial (16.7) is rewritten as follows:

$$\tilde{F}(j\omega) = [P^-(\omega), P^+(\omega)] + j[Q^-(\omega), Q^+(\omega)].$$

Therefore, four corner points for some ω are given as

$$\boldsymbol{V}_1 = (P^+, Q^+), \quad \boldsymbol{V}_2 = (P^-, Q^-)$$
$$\boldsymbol{V}_3 = (P^-, Q^+), \quad \boldsymbol{V}_4 = (P^+, Q^-),$$

and the Kharitonov rectangle is drawn as shown in Fig. 16.1.

The necessity of the statement is clear, because the four corner points are included in the rectangle. On the other hand, the sufficiency is proved as a simple case of Theorem 6.1. The proof is given from the zero exclusion of the Kharitonov rectangle. That is, none of the edges of the rectangle pass through the origin.

(7) Show that the Routh series and Hurwitz deteriminant can be derived directly from

$$F(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

by applying the division algorithm to

$$\begin{cases} f_0(s) = a_0 s^n + a_2 s^{n-2} + \cdots \\ f_1(s) = a_1 s^{n-1} + a_3 s^{n-3} + \cdots . \end{cases}$$

Ans. The direct division process for $f_0(s)/f_1(s)$ is given as follows:

$$
\begin{array}{ll}
 & (a_0/a_1)s \\
a_1 s^{n-1} + a_3 s^{n-3} + a_5 s^{n-5} + \cdots & \big)\overline{a_0 s^n + a_2 s^{n-2} \;+\; a_4 s^{n-4} \;+ \cdots\cdots} \\
 & \underline{\;a_0 s^n + (a_0 a_3/a_1)s^{n-2} + (a_0 a_5/a_1)s^{n-4} + \cdots} \\
 & \qquad\quad a_{2,2}s^{n-2} \;+\; a_{2,4}s^{n-4} + \cdots ,
\end{array}
$$

where

$$a_{2,2} = a_2 - (a_0 a_3/a_1), \quad a_{2,4} = a_4 - (a_0 a_5/a_1), \quad \cdots .$$

Next,

$$
\begin{array}{ll}
 & (a_1/a_{2,2})s \\
a_{2,2}s^{n-2} + a_{2,4}s^{n-4} + \cdots & \big)\overline{a_1 s^{n-1} \;+\; a_3 s^{n-3} + a_5 s^{n-5} \;+ \cdots\cdots\cdots} \\
 & \underline{\;a_1 s^{n-1} + (a_1 a_{2,4}/a_{2,2})s^{n-3} + (a_1 a_{2,6}/a_{2,2})s^{n-5} + \cdots} \\
 & \qquad\qquad a_{3,3}s^{n-3} \;+\; a_{3,5}s^{n-5} + \cdots\cdots \quad ,
\end{array}
$$

where

$$a_{3,3} = a_3 - (a_1 a_{2,4}/a_{2,2}), \quad a_{3,5} = a_5 - (a_1 a_{2,6}/a_{2,2}), \quad \cdots .$$

Moreover,

$$
\begin{array}{ll}
 & (a_{2,2}/a_{3,3})s \\
a_{3,3}s^{n-3} + a_{3,5}s^{n-5} + \cdots & \big)\overline{a_{2,2}s^{n-2} \;+\; a_{2,4}s^{n-4} \;+\; a_{2,6}s^{n-6} + \cdots} \\
 & \underline{\;a_{2,2}s^{n-2} \;+\; (a_{2,2}a_{3,5}/a_{3,3})s^{n-4} + \cdots} \\
 & \qquad\qquad a_{4,4}s^{n-4} \;+\; a_{4,6}s^{n-6} + \cdots \quad ,
\end{array}
$$

where

$$a_{4,4} = a_{2,4} - (a_{2,2}a_{3,5}/a_{3,3}), \quad a_{4,6} = a_{2,6} - (a_{2,2}a_{3,7}/a_{3,3}), \quad \cdots .$$

Here, the tops of these polynomials correspond to Routh's series, and are rewritten as follows:

$$
\begin{aligned}
a_{2,2} &= \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} \Big/ a_1 \\
a_{3,3} &= \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} \Big/ \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} \\
a_{4,4} &= \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix} \Big/ \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} \\
&\cdots \qquad\qquad (16.8) \\
a_{n,n} &= a_n, \quad (\text{because } \ a_m = 0 \ \text{ for } \ m > n).
\end{aligned}
$$

As was shown in (6.94), each term of (16.8) corresponds to a Hurwitz determinant and its principal minors.

Chapter 7 Relation to Discrete Event Systems

(1) Draw the state transition graph for a vending machine with four states and two events. Show the events and states trajectories.

Ans. The set of states will be defined as

$$\mathcal{X} = \{0, 1, 2, 3\},$$

where $x = 0, 1, 2, 3$ denote the stored amount in the machine by coins (a **1** or **2**). Here, let us consider that the set of events is given by

$$\mathcal{E} = \{e^1, e^2\},$$

where e^1 (e^2) denotes a **1** (**2**) euro coin being inserted into the coin slot. The state transition graph is as shown in Fig. 17.2. In the figure, e^2/**1** indicates an event in which a **1** euro coin is returned as change. The sets of event trajectories are written as

$$\begin{cases} (1)\ \{e^1, e^1, e^1, e^1\},\ \{e^1, e^1, e^2\},\ \{e^1, e^2, e^1\},\ \{e^2, e^1, e^1\},\ \{e^2, e^2\} \\ (2)\ \{e^1, e^1, e^1, e^2\},\ \{e^1, e^2, e^2\},\ \{e^2, e^1, e^2\}. \end{cases}$$

Here, in the case of (2), one euro will be returned.

The above correction in the question (three $\rightarrow$ two) is based on that there is no **3** euro coin. If (fictitious) **3** euro coin is considered, the set of events is given as

$$\mathcal{E} = \{e^1, e^2, e^3\},$$

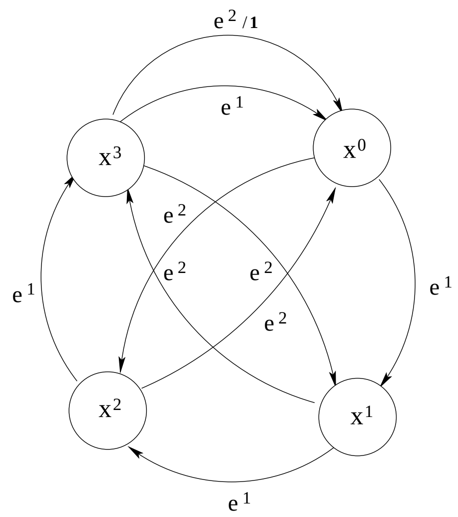

Fig. 17.2 Vending machine for question (1).

where e^1 (e^2 or e^3) denotes a **1** (**2** or **3**) coin being inserted into the coin slot. Therefore, the sets of event trajectories become as follows:

$$\begin{cases} (1)\ \{e^1,e^1,e^1,e^1\},\ \{e^1,e^1,e^2\},\ \{e^1,e^2,e^1\},\ \{e^2,e^1,e^1\},\ \{e^1,e^3\},\ \{e^2,e^2\},\ \{e^3,e^1\}, \\ (2)\ \{e^1,e^1,e^1,e^2\},\ \{e^1,e^1,e^3\},\ \{e^1,e^2,e^2\},\ \{e^2,e^1,e^2\},\ \{e^2,e^3\},\ \{e^3,e^2\}, \\ (3)\ \{e^1,e^1,e^1,e^3\},\ \{e^1,e^2,e^3\},\ \{e^2,e^1,e^3\},\ \{e^3,e^3\}. \end{cases}$$

Obviously, in case (2) one euro and in case (3) two euro will be returned as change.

(2) Consider a machine with two buffers that can only process one part of one type at a time.

(i) Define the set of states and events.

(ii) Draw the state transition graph for the buffer machine.

Ans.

(i) Each set of states of the buffers is assigned as shown in (7.9),

$$\begin{cases} \mathcal{X}_1 = \{x_1^0,\ x_1^1,\ x_1^2,\ \cdots\} = \{0,1,2,3,\cdots\} \\ \mathcal{X}_2 = \{x_2^0,\ x_2^1,\ x_2^2,\ \cdots\} = \{0,1,2,3,\cdots\}. \end{cases}$$

First, consider the case of a series connection of buffers as shown in Fig. 17.3 (a). In this case, the set of events are given by $\mathcal{E} = \{e^1, e^2, e^3, e^4, e^5\}$, where

- e^1 = "a part arrives in buffer 1"
- e^2 = "buffer 2 has received a part from buffer 1"

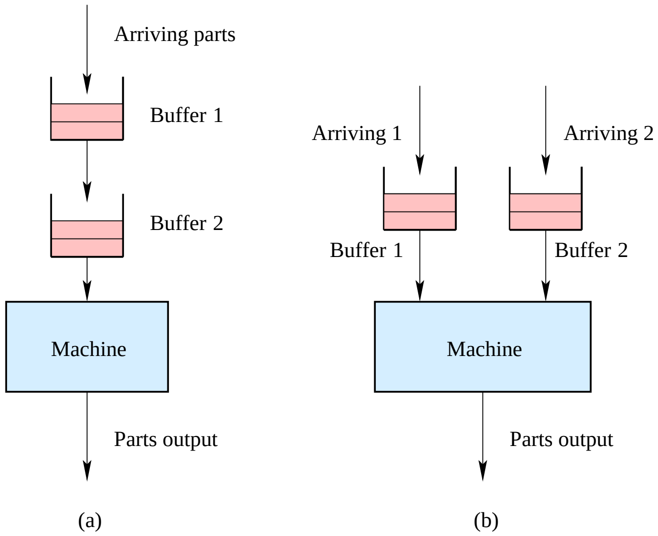

Fig. 17.3 Machine with two buffers.

- e^3 = "buffer 2 has received a part *and* a part arrives in buffer 1 at the same time"
- e^4 = "the machine has finished processing a part"
- e^5 = "the machine has finished processing a part *and* buffer 2 receives a part at the same time".

(ii) The state transition graph is as shown in Fig. 17.4 (a). If a pair of the number of parts is defined as a state of the buffers, the state transition graph is as shown in Fig. 17.4 (b). The set of states of the buffers is assigned as follows:

$$\begin{aligned}\mathcal{X} &= \{x^0, x^1, x^2, x^3, x^4, x^5\, x^6, x^7, x^8\}\\ &= \{(0,0), (1,0), (2,0), (0,1), (1,1), (2,1), (0,2), (1,2), (2,2)\}.\end{aligned}$$

For simplicity, the number of parts in a buffer is restricted to 2 parts.

(1)' Next, consider the case of a parallel connection of buffers as shown in Fig. 17.3 (b). In this case, the set of events are given by $\mathcal{E} = \{e^1, e^2, e^3, e^4, e^5, e^6\}$, where

- e^1 = "a part arrives in buffer 1"
- e^2 = "a part arrives in buffer 2"
- e^3 = "the machine has finished processing a part from buffer 1"
- e^4 = "the machine has finished processing a part from buffer 2"
- e^5 = "the machine has finished processing a part *and* buffer 1 receives a part at the same time".
- e^6 = "the machine has finished processing a part *and* buffer 1 receives a part at the same time".

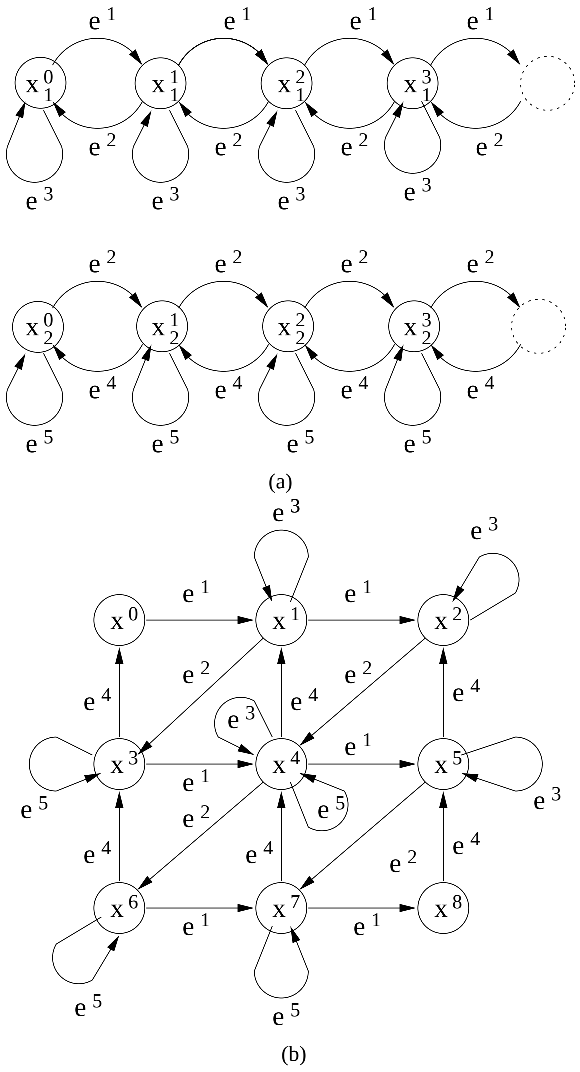

Fig. 17.4 State transition graph for question (2).

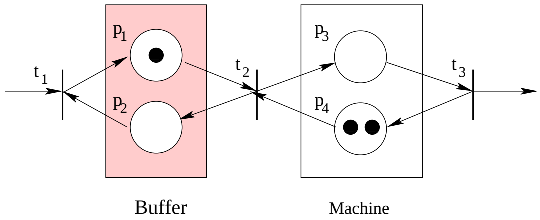

Fig. 17.5 Petri net graph for question (3).

(ii)' The state transition graphs that correspond to Figs. 17.4 (a) and (b) are left for the reader.

(3) Produce a Petri net model of the single buffer machine that is modeled in Sect. 7.3.2.

Ans. An example of the production network with a buffer is given as shown in Fig. 17.5. Here, p_1 and p_2 are places in Buffer, and p_3 and p_4 are places in Machine. Moreover,

- $\mathcal{M}(p_1)$ is the number of parts in Buffer,
- $\mathcal{M}(p_2)$ is a part counter for the buffer,
- $\mathcal{M}(p_3)$ is the number of parts being processed by Machine.
- $\mathcal{M}(p_4)$ is the number of parts waiting to be processed by the machine,

(4) Show a candidate of Lyapunov (nonincreasing and non-negative) function in Sect. 7.6.

Ans. There are some expressions for the multiple Lyapunov function. Here, consider the following multiple metric:

$$\boldsymbol{V}(\boldsymbol{x}) := \begin{bmatrix} V_1(\boldsymbol{x}) \\ V_2(\boldsymbol{x}) \\ \vdots \\ V_n(\boldsymbol{x}) \end{bmatrix},$$

where $V_i(\boldsymbol{x})$ $(i = 1, 2, \cdots, n)$ is defined as, e.g.,

$$V_i(\boldsymbol{x}) := \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x},$$

that is, a quadratic form (See Appendix A in Chapter 1). When $n = 3$, an example of the Liapunov function is given as

$$V_i(\boldsymbol{x}) = ax_1^2 + bx_2^2 + cx^2, \quad a > 0, \ \ b > 0, \ \ c > 0.$$

Furthermore, when $\boldsymbol{A}_i$ is a singular matrix, the following functions can be defined in the special case:

$$V_i(\boldsymbol{x}) = x_1^2 + x_2^2$$
$$V_i(\boldsymbol{x}) = x_1^2.$$

If the elapsed time t_k is considered, the following multiple metric is defined based on the above Lyapunov function:

$$\|\boldsymbol{V}(\boldsymbol{x}(t_k))\|_{\ell_1} := \begin{bmatrix} \|V_1(\boldsymbol{x}(t_k))\|_{\ell_1} \\ \|V_2(\boldsymbol{x}(t_k))\|_{\ell_1} \\ \vdots \\ \|V_n(\boldsymbol{x}(t_k))\|_{\ell_1} \end{bmatrix},$$

where

$$\|V_i(\boldsymbol{x}(t_k))\|_{\ell_1} = \sum_{k=0}^{\infty} V_i(\boldsymbol{x}(t_k)).$$

(5) Describe the difference between sectors (7.35) and (7.37).

Ans. From (7.31), the definition of f_i is written as

$$f_i(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)) = f_{i1}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)) + f_{i2}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)) + \cdots + f_{in}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)).$$

Here, each term f_{i1}, f_{i2}, $\cdots$, f_{in} can be arbitrarily partitioned. Therefore, if inequality (7.35), i.e.,

$$|\psi_{ij}(t_k)| = \frac{|f_{ij}(x, e)|}{|x_j(t_k)|} \leq \bar{\psi}_{ij}$$

is determined, $\bar{\psi}_{ij}$ corresponds to a sector of transmission $x_j \to f_i$. On the other hand, if inequality (7.37), i.e.,

$$|\psi_{ii}(t_k)| = \frac{|f_i(x, e)|}{|x_j(t_k)|} \leq \bar{\psi}_i$$

can only be determined, $\bar{\psi}_{ij}$ becomes the sector of transmission $x_i \to f_i$.

(6) Show that the matrix condition (7.41) is invariant, when considering (7.39) in the ℓ_∞ space, i.e.,

$$\|x_i(t_k)\|_\infty = \sup_{k \in \mathbb{Z}} |x_i(t_k)|$$

and

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} = \begin{bmatrix} \|x_1(t_k)\|_\infty \\ \|x_2(t_k)\|_\infty \\ \vdots \\ \|x_n(t_k)\|_\infty \end{bmatrix}.$$

Ans. From (7.30), the following inequality can be obtained:

$$\begin{aligned} \|\boldsymbol{x}\|_{\ell_\infty} &\leqq \|\boldsymbol{\Phi}(t_k, t_0)\boldsymbol{x}(t_0)\|_{\ell_\infty} + \left\| \sum_{l=1}^{k} \boldsymbol{\Phi}(t_k, t_l)\boldsymbol{\Psi}(t_{l-1})\boldsymbol{x}(t_{l-1}) \right\|_{\ell_\infty} \\ &\leqq \|\boldsymbol{\Phi}(t_k, t_0)\|_{\ell_\infty} |\boldsymbol{x}(t_0)| + \left(\sum_{l=1}^{k} |\boldsymbol{\Phi}(t_k, t_l)| \right) \bar{\boldsymbol{\Psi}} \cdot \|\boldsymbol{x}(t_k)\|_{\ell_\infty}. \end{aligned}$$

Therefore,

$$\left[\boldsymbol{I} - \left(\sum_{l=1}^{k} |\boldsymbol{\Phi}(t_k, t_l)|\right) \bar{\boldsymbol{\Psi}}\right] \|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\boldsymbol{\Phi}(t_k, t_0)\|_{\ell_\infty} |\boldsymbol{x}(t_0)|.$$

In consequence, the stability condition is that (7.41) should be Ostrowski's M-matrix.

(7) Show that the vector-matrix expression with non-negative elements as shown in Appendix B can also defined for multiple metrics or Lyapunov functions.

Ans. The vector-matrix expresion of Lyapunov function was given in question (4). With respect to the ℓ_∞ space, the multiple metric is, for example, defined as follows:

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} = \begin{bmatrix} \|x_1(t_k)\|_{\ell_\infty} \\ \|x_2(t_k)\|_{\ell_\infty} \\ \vdots \\ \|x_n(t_k)\|_{\ell_\infty} \end{bmatrix},$$

where

$$\|x_i(t_k)\|_{\ell_\infty} = \sup_{k \in \mathbb{Z}} |x_i(t_k)|.$$

For matrix expression, the following can be defined:

$$\|\boldsymbol{\Psi}(t_k)\|_{\ell_\infty} = \begin{bmatrix} \sup_{k\in\mathbb{Z}} |\psi_{11}(t_k)| & \sup_{k\in\mathbb{Z}} |\psi_{12}(t_k)| & \dots & \sup_{k\in\mathbb{Z}} |\psi_{1n}(t_k)| \\ \sup_{k\in\mathbb{Z}} |\psi_{21}(t_k)| & \sup_{k\in\mathbb{Z}} |\psi_{22}(t_k)| & \dots & \sup_{k\in\mathbb{Z}} |\psi_{2n}(t_k)| \\ \vdots & \vdots & \ddots & \vdots \\ \sup_{k\in\mathbb{Z}} |\psi_{n1}(t_k)| & \sup_{k\in\mathbb{Z}} |\psi_{n2}(t_k)| & \dots & \sup_{k\in\mathbb{Z}} |\psi_{nn}(t_k)| \end{bmatrix}.$$