Stability Analysis of Event-Driven Control Systems on Lattice Coordinates

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Abstract– Nowadays there are many event-driven types of discrete control systems in practice, e.g., manufacturing automation systems, industrial and welfare robots, networked control systems, and so on. Therefore, in this paper, the (finite-time) stability of event-driven (in other words, event-based) control systems is studied. First, the lattice concept based on ordered sets is reviewed in general and the graphical representations of discrete event systems are compared. Then, the (technological) state transitions are considered on (integer) lattice/grid coordinates. Lastly, the stability of such event-driven discrete systems is analyzed using multiple metrics and simultaneous linear inequalities. Numerical examples are shown to clarify the stability and boundedness of event-driven control systems.

Key Words: Discrete event systems, Digitally networked systems, Multiple metrics, Nonnegative Inverse matrices, Relative stability

1 Introduction

At present, there are many event-driven types of discrete systems in practice, e.g., manufacturing systems, industrial and welfare robots, computer networked systems, and so forth. Therefore, in this paper, the (finite-time) stability of event-driven (in other words, event-based) control systems is studied. First, the lattice concept based on ordered sets is reviewed in general and the graphical representations of event-driven type of discrete systems are compared. Especially, in this paper, the topological representations of discrete systems will be regarded as important. Then, the (technological) state transitions are considered on (integer) lattice/grid coordinates. Lastly, the stability of such event-driven discrete systems is analyzed using multiple metrics and simultaneous linear inequalities.

Incidentally, in order to avoid becoming complicated, most of the event-driven discrete systems considered here will be restricted to third-order ones.

2 Ordered Sets and Lattices

Suppose a binary relation R on a given set E satisfying the following properties:

(1) Reflexive $(\forall x \in E)\ (x, x) \in R$.

(2) Anti-symmetric $(\forall x, y \in E)$ if $(x, y) \in R$ and $(y, x) \in R$ then $x = y$.

(3) Transitive $\forall x, y, z \in E$ if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

In these descriptions, whenever $(x, y) \in R$, $x, y \in E$ are referred to as 'R-related' and often written as xRy[1, 2)].

Based on the above thoughts, R is called a *partial order* or, simply, an *order* relation. Moreover, the set E with the partial order is called a *partially ordered set* or, simply, '*poset*'. When specifying the relation R we will use the expression $(E; R)$.

The word 'partial' is used in defining a partially ordered set E since some of the elements of E need not be comparable. On the other hand, if any two elements of E are comparable, E is said to be *totally ordered* or *linearly ordered*, and E is called a *chain* [1].

An example of the quantitative relation is $\leq$ ("less than or equal to") which is regarded as the *usual order*. In this case, the above properties are written as[1, 4)]:

(1) $(\forall x \in E)\ x \leq x$.

(2) $(\forall x, y \in E)$ if $x \leq y$ and $y \leq x$ then $x = y$.

(3) $(\forall x, y, z \in E)$ if $x \leq y$ and $y \leq z$ then $x \leq z$.

When using the expression $(E; R)$, the above relation can be specified as $(E; \leq)$. Of course, with respsect to the relation $\geq$ ("greater than or equal to") the similar description can be given.

On the other hand, in regard to the qualitative relation the following symbols will be used:

$$\preceq, \quad \succeq$$

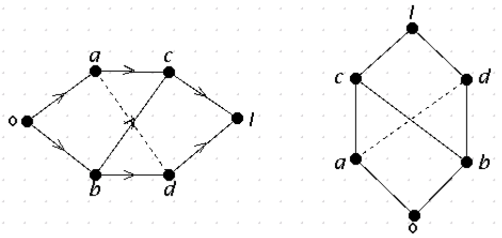
instead of $\leq$, $\geq$. Here, $x \preceq y$, $x \succeq y$ are read "x precedes y" and "x succeeds y", respectively[2)].

From the above premise, we can define a *lattice* as follows.

Definition.
A lattice is an ordered set L in which every pair of elements (and hence every finite subset) has an infimum (meet) and a supremum (join). Thus we often denote a lattice by $(L; \wedge, \vee)$ or $(L; \wedge, \vee, \leq)$. □

In mathematics, 'lattice' is translated into Japanese as "soku". The author thinks that the translation is not appropriate. A lattice means originally wood frames ("koushi") of the door or window, which are found in Japanese traditional house. In the previous papers, the author has used 'grid' for grid frame coordinates. A 'grid' means originally metallic ones (e.g., screen-grid in an electron tube). However, each of them is translated into Japanese as the same word, "koushi". In this paper, 'lattice' will be used in the mathematical meaning [1, 3, 4)].

[1]The sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ of natural numbers, integers, rationals, and real numbers form chains under their usual orders.



(a) (b)

Fig. 1: Hasse diagram.

3 Graph Representations

3.1 Directed Graph and Hasse Diagram

The structure of lattice is easy to understand by using some graphic representations. Figure 1 shows examples of lattice with six vertices. The left of the figure, (a), is drawn based on the directed graph (using directed edges)[2]. However, we sometimes place c higher than a and draw a line (without an arrow) between them as shown in (b). It is then understood that an upward movement indicates *succession* (otherwise *precession*). Such a graph representation is refer to as the *Hasse diagram*[1, 2, 3, 4, 5)].

3.2 Matrices and Tables

A matrix and also a table will represent the system structure clearly. The matrix is translated into Japanese as "gyoretsu". However, it is known that the word 'matrix' is originated from "botai", really 'mother'. Thus, the translation in Japanese would not be always appropriate since "gyoretsu" in Japanese corresponds to 'queue' in English.

In the first half of this paper, since only system structures are considered, we assume that matrices are simply given as follows:

$$\boldsymbol{A} \in \mathbb{I}^{n\times n} \subseteq \mathbb{Z}^{n\times n}, \quad \mathbb{I} = \{-1, 0, 1\}.$$

Moreover, to avoid becoming complicated, only third-order discrete systems are considered. The matrix expressions, for example, are given as follows:

$$\boldsymbol{A}_1 = \begin{bmatrix} 0^* & 0 & 1 \\ 1 & 0^* & 1 \\ 1 & 0 & {}^*0 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 0^* & 1 & 0 \\ 0 & 0^* & 1 \\ 1 & 1 & {}^*0 \end{bmatrix},$$

Here, the symbol 0^* denotes 0 or 1. When there exists a self-loop as shown in a signal flow graph, it become '1' . Of course, in the linear algebraic expression it may be any real number.

On the other hand, the tabular expressions are given as Table 1.

[2]A *directed graph* (*digraph*) is written by $G(V, E)$. Here, V are called *vertices*, *nodes*, or *points*, and E are called *arcs* or *directed edges* or simply *edges* [6, 7)]. It should be noted that an *oriented graph* is a directed graph which has no symmetric pair of directed edges. However, either of them translated into Japanese "yuko"graph or "teiko" graph.

(a) (b)

Fig. 2: Matrices and directed graph.

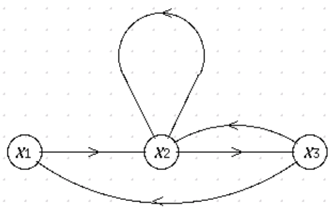


Fig. 3: Signal flow graph.

Table 1: Relation between matrices and table.

(a)

	x_1	x_2	x_3
x_1	0^*	0	1
x_2	1	0^*	1
x_3	1	0	0^*

(b)

	x_1	x_2	x_3
x_1	0^*	0	0
x_2	1	0^*	1
x_3	1	1	0^*

When using the directed graph expression they can be drawn as shown in Fig. 2

3.3 Linear Algebra and Signal Flow Graph

When considering the linear algebra and the state trasmission, the following transposed matrices are used:

$$\boldsymbol{A}_1^T = \begin{bmatrix} 0^* & 1 & 1 \\ 0 & 0^* & 0 \\ 1 & 1 & 0^* \end{bmatrix}, \quad \boldsymbol{A}_2^T = \begin{bmatrix} 0^* & 0 & 1 \\ 1 & 0^* & 1 \\ 0 & 1 & {}^*0 \end{bmatrix},$$

With respect to $\boldsymbol{A}_2^T$ we assume that there exists a self-loop for x_2. In this case, the state transmission can be expressed as:

$$\begin{bmatrix} x_1(t_{k+1}) \\ x_2(t_{k+1}) \\ x_3(t_{k+1}) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1(t_k) \\ x_2(t_k) \\ x_3(t_k) \end{bmatrix}$$
$$k = 0, 1, 2, \ldots.$$

Therefore, when we use the signal-flow-graph representation, the above equation can be expressed as shown in Fig. 3.

4 Lattice Coordinates

In regard to the usual orthogonal coordinates (parallel coordinates in general), the following lattice properties will be satisfied[1, 2, 3, 4, 5)]:

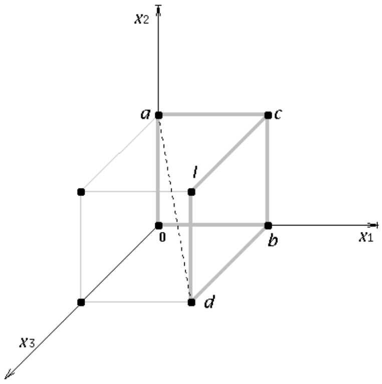


Fig. 4: A basic 3D lattice.

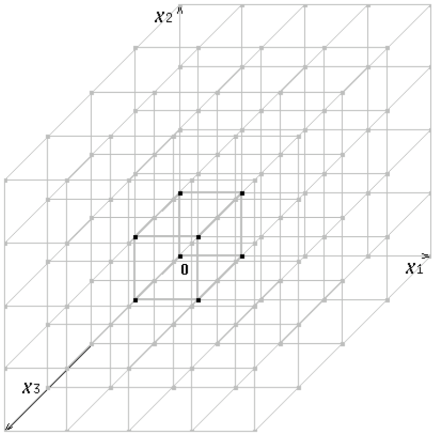


Fig. 5: An example of 3D lattice.

4.1 Distributive and Modular Lattices

(1) L is said to be *distributive* if it satisfies the distributive law

$$(\forall x, y, z \in L) \quad z \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

(2) L is said to be *modular* if it satisfies the modular law

$$(\forall x, y, z \in L) \quad x \geq z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z.$$

4.2 Integer Lattice Coordinates

Even if states and events are qualitative ordered sets, we can replace them with (positive) integer numbers (i.e., a chain) as follows:

$$\begin{array}{ccccccccc} x_i^0 & \prec & x_i^1 & \prec & x_i^2 \prec & \cdots & \prec & x_i^N \\ \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ 0 & & 1 & & 2 & \cdots & & N. \end{array}$$

Of course, the above correspondence can be expanded to negative numbers. Figures 4 and 5 show 3D expressions of lattices. Obviously, Fig 4 will correspond to a boolean lattice[4)]. It should be noted that the subgraph drawn by bold lines corresponds to Fig. 1 (b). In these figures. the coordinates is drawn only for the first quadrant.

Incidentally, it should be noted that the fundamental properties of lattice is lost by the existence of dotted lines in Figs. 1 (a), (b), and Fig. 4.

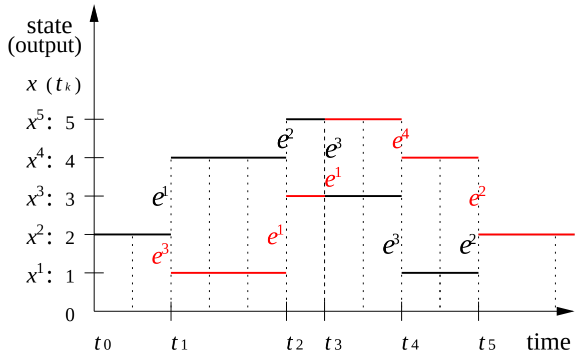


Fig. 6: State trajectories.

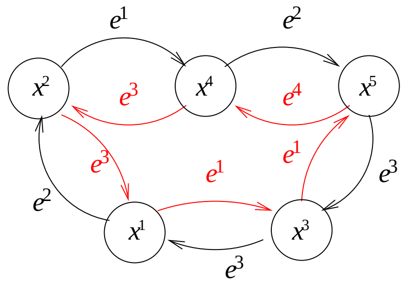


Fig. 7: State transition graph.

5 Event-Driven Discrete Systems and State Transition

Figure 6 shows an example of state (or output) trajectories of a first-order (event-driven) discrete system. As is obvious from the figure, the event-sequence $\{e^1e^2e^3e^3e^2\cdots\}$ ($\{e^3e^1e^1e^4e^3\cdots\}$) corresponds to time sequence $\{t_1t_2t_3t_4t_5\cdots\}$. However, the causality relationship between them will be opposite. Figure 7 shows their state transition graphs.

In general, event-driven types of discrete systems can be written as

$$\begin{aligned} &\boldsymbol{x}(t_{k+1}) = \boldsymbol{f}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)) \\ &k = 0, 1, 2, \ldots. \end{aligned}$$

As is obvious from the figure, the time sequence $t_0, t_1, t_2, \ldots$ is not periodic for these event-driven systems. However, they can also correspond to integer numbers.

Here, in order to study the relative stability problem, we consider the following semi-linear discrete systems.

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{\Phi}(t_{k+1}, t_k)\boldsymbol{x}(t_k) + \boldsymbol{f}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)). \quad (1)$$

The transition matrix $\boldsymbol{\Phi}(\cdot, \cdot)$ is considered time-invariant and written as

$$\boldsymbol{\mathcal{P}} := \boldsymbol{\Phi}(t_{k+1}, t_k) \in \mathbb{Z}^{n \times n}, \qquad \forall k \in \mathbb{N}. \quad (2)$$

If we simplify it only system structure, the transition matrix may be written as:

$$\boldsymbol{\mathcal{P}} \in \mathbb{I}^{n \times n} \subseteq \mathbb{Z}^{n \times n}, \quad \mathbb{I} := \{-1, 0, 1\}. \quad (3)$$

In regard to time-invariant nominal systems, the following will be valid from (2):

(1) $\boldsymbol{\Phi}(t_k, t_l) = \boldsymbol{\mathcal{P}}^{k-l}$,

(2) $\boldsymbol{\Phi}(t_k, t_l) = \boldsymbol{\mathcal{P}}^0 = \boldsymbol{I}$.

Thus, when the nominal system is periodic, the following property can be obtained:

(3) $\boldsymbol{\Phi}(t_{k+p}, t_k) = \boldsymbol{\mathcal{P}}^p = \boldsymbol{I}, \quad p \in \mathbb{N}$: period.

By expanding (1), the following equation is obtained:

$$\boldsymbol{x}(t_k) = \boldsymbol{\Phi}(t_k, t_0)\boldsymbol{x}(t_0) + \sum_{l=1}^{k} \boldsymbol{\Phi}(t_k, t_l)\boldsymbol{f}(\boldsymbol{x}(t_{l-1}), \boldsymbol{e}(t_{l-1})). \tag{4}$$

Using (2), it can also be written as follows:

$$\boldsymbol{x}(t_k) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0) + \sum_{l=1}^{k} \boldsymbol{\mathcal{P}}^{k-l} \boldsymbol{f}(\boldsymbol{x}(t_{l-1}, \boldsymbol{e}(t_{l-1}))).$$
$$k = 0, 1, 2, \ldots. \tag{5}$$

In any case, the nominal system can be given as follows:

$$\bar{\boldsymbol{x}}(t_k) = \boldsymbol{\Phi}(t_k, t_0)\boldsymbol{x}(t_0) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0) \in \mathbb{Z}^n. \tag{6}$$

In this paper, the event driven function $\boldsymbol{f}$ is simplified as

$$\varepsilon_{ii}(t_k) = \frac{f_i(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k))}{x_i(t_k)} \in \mathbb{R}. \tag{7}$$

In regard to a diagonal matrix, the following expression can be written:

$$\boldsymbol{\mathcal{E}}(t_k) = \begin{bmatrix} \varepsilon_{11}(t_k) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon_{nn}(t_k) \end{bmatrix}. \tag{8}$$

Based on the above premises, (5) can be written as:

$$\boldsymbol{x}(t_k) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0) + \sum_{l=1}^{k} \boldsymbol{\mathcal{P}}^{k-l} \boldsymbol{\mathcal{E}}(t_{l-1})\boldsymbol{x}(t_{l-1}). \tag{9}$$

Here, let us consider a new type of transition matrix, i.e.,

$$\boldsymbol{\Psi}(k, l) := \boldsymbol{\mathcal{P}}^{k-l} \boldsymbol{\mathcal{E}}(t_{l-1}).$$

Thus, (9) can be rewritten as follows:

$$\boldsymbol{x}(t_k) = \bar{\boldsymbol{x}}(t_k) + \sum_{l=1}^{k} \boldsymbol{\Psi}(k, l)\boldsymbol{x}(t_{l-1}), \tag{10}$$

where $\bar{\boldsymbol{x}}(t_k) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0)$ is the nominal state response.

6 Multiple Metrics and Inequalities

The metric in the state space (i.e., vector space) is usually defined by a scalar value. However, it may lead to a severe condition for the stability of some kind of nonlinear systems. Therefore, we consider the metric (i.e., norm) for each element of the state as follows:

$$\|x_i(t_k)\|_{\ell_\infty} := \sup_{1 \le l \le k} |x_i(t_l)| \in \mathbb{Z}_+ \tag{11}$$

or simply the absolute value,

$$\|x_i(t_k)\|_{\ell_\infty} := |x_i(t_k)| \in \mathbb{Z}_+. \tag{12}$$

When considering multiple metrics, the following vector can be written:

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} = \begin{bmatrix} \|x_1(t_k)\|_{\ell_\infty} \\ \|x_2(t_k)\|_{\ell_\infty} \\ \vdots \\ \|x_n(t_k)\|_{\ell_\infty} \end{bmatrix} \in \mathbb{Z}^n. \tag{13}$$

Based on these considerations, the following inequalities are obtained from (10):

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty} + \left\| \sum_{l=1}^{k} \boldsymbol{\Psi}(k, l)\boldsymbol{x}(t_{l-1}) \right\|_{\ell_\infty}. \tag{14}$$

Here, we define a matrix expression with positive elements,

$$\|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} = \begin{bmatrix} \|\theta_{11}(t_k)\|_{\ell_\infty} & \ldots & \|\theta_{1n}(t_k)\|_{\ell_\infty} \\ \vdots & \ddots & \vdots \\ \|\theta_{n1}(t_k)\|_{\ell_\infty} & \ldots & \|\theta_{nn}(t_k)\|_{\ell_\infty} \end{bmatrix}, \tag{15}$$

where,

$$\|\theta_{ij}(t_k)\|_{\ell_\infty} := \left\| \sum_{l=1}^{k} \psi_{ij}(k, l) x_j(t_{l-1}) \right\|_{\ell_\infty} / \|x_j(t_k)\|_{\ell_\infty}$$
$$i, j = 1, 2, \cdots, n. \tag{16}$$

Therefore, inequality (14) can be written as:

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty} + \|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} \cdot \|x(t_k)\|_{\ell_\infty}. \tag{17}$$

Moreover, it can be written as follows:

$$\left(\boldsymbol{I} - \left\|\boldsymbol{\Theta}(t_k)\right\|_{\ell_\infty}\right) \|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty}. \tag{18}$$

7 Nonnegative-Inverse Matrix and Stability Conditions

By using the above inequality expressions, the stability analysis of event-driven discrete systems is given as follows:

Definition.
If $\|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty} \leqq \bar{\boldsymbol{X}}$ leads to $\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \boldsymbol{X}$ for all $k \in \mathbb{N}$, the discrete event system is defined as (finite-time) stable in a relative sense [8]. Here, $\boldsymbol{X}$ and $\bar{\boldsymbol{X}}$ are vectors of some finite (positive) numbers. □

Thus, the following theorem is given.

Theorem 1.
If there exists a vector $\boldsymbol{0} \leqq \boldsymbol{X} < \infty$ by which the

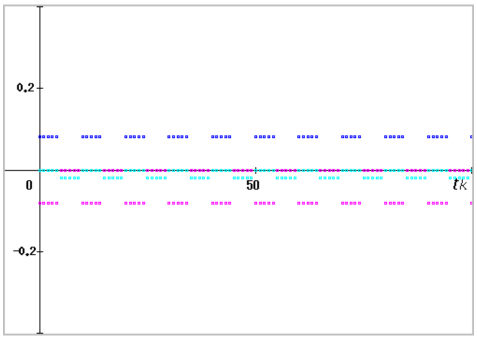


Fig. 8: Time series of event signals, $e_1 = 0.08$, $e_2 = -0.08$, and $e_3 = -0.02$.

following equation holds in regard to a vector $\bar{\boldsymbol{X}}$ with (bounded) positive elements:

$$(\boldsymbol{I} - \|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} \boldsymbol{X} \leqq \bar{\boldsymbol{X}}, \tag{19}$$

the discrete event system is (finite-time) stable in a relative sense.

In other words, the matrix of the left side of (19), i.e.,

$$\boldsymbol{\mathcal{A}}(t_k) = \boldsymbol{I} - \|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} \tag{20}$$

is a nonnegative-inverse matrix[9] (i.e, Ostrowski's M-matrix[10]), the system becomes (finite-time) stable in a relative sense.

Proof. If (20) is a nonnegative-inverse matrix, (17) can be written as follows:

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \left(\boldsymbol{I} - \left\|\boldsymbol{\Theta}(t_k)\right\|_{\ell_\infty}\right)^{-1} \|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty} < \infty. \tag{21}$$

Therefore, the relative stability of event-driven systems (9) and (10) has been proved. □

8 Numerical Examples

Example 1. First, consider the following recurrent third-order system:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k + \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}_k \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k . \tag{22}$$

The structure matrix is given below:

$$\boldsymbol{\mathcal{P}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\mathcal{P}}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \boldsymbol{\mathcal{P}}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Obviously, the nominal system of this example contains a periodic mode with $p = 3$. Assumed that event-driven signals e_1, e_2, and e_3 are as shown in Fig. 8. Figs. 9 and 10 are state trajectories from different initial conditions. The former is unstable (divergent) case and the latter is stable (bounded) pseudo-periodic case.

From (15), $\boldsymbol{A}(t_k)$ in Theorem 1 can be written as:

$$\boldsymbol{A}(t_k) = \boldsymbol{I} - \|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} =$$

$$\begin{bmatrix} 1 - \|\theta_{11}(t_k)\|_{\ell_\infty} & -\|\theta_{12}(t_k)\|_{\ell_\infty} & -\|\theta_{13}(t_k)\|_{\ell_\infty} \\ -\|\theta_{21}(t_k)\|_{\ell_\infty} & 1 - \|\theta_{22}(t_k)\|_{\ell_\infty} & -\|\theta_{23}(t_k)\|_{\ell_\infty} \\ -\|\theta_{31}(t_k)\|_{\ell_\infty} & -\|\theta_{32}(t_k)\|_{\ell_\infty} & 1 - \|\theta_{33}(t_k)\|_{\ell_\infty} \end{bmatrix}.$$

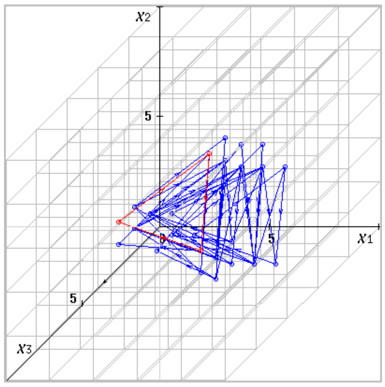


Fig. 9: State trajectories in 3D phase space when $x_1(0) = 3.0$, $x_2(0) = 4.0$, and $x_3(0) = 1.0$).

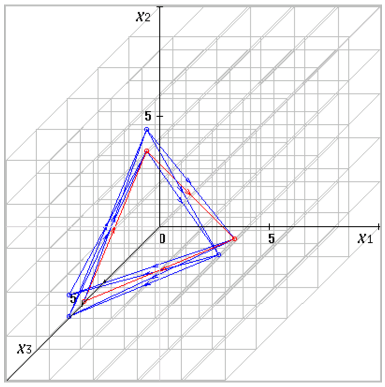


Fig. 10: State trajectories in 3D phase space when $x_1(0) = -2.0$, $x_2(0) = 2.0$, and $x_3(0) = -2.0$).

Therefore, the stability condition is given below[11, 12, 13, 14]:

$$\begin{cases} \Delta_1 = 1 - \|\theta_{11}(t_k)\|_{\ell_\infty} > 0 \\ \Delta_2 = (1 - \|\theta_{11}(t_k)\|_{\ell_\infty})(1 - \|\theta_{22}(t_k)\|_{\ell_\infty}) \\ \quad - \|\theta_{12}(t_k)\|_{\ell_\infty}\|\theta_{21}(t_k)\|_{\ell_\infty} > 0 \\ \Delta_3 = (1 - \|\theta_{11}(t_k)\|_{\ell_\infty})(1 - \|\theta_{22}(t_k)\|_{\ell_\infty})(1 - \|\theta_{33}(t_k)\|_{\ell_\infty}) \\ \quad - \|\theta_{12}(t_k)\|_{\ell_\infty}\|\theta_{23}(t_k)\|_{\ell_\infty}\|\theta_{31}(t_k)\|_{\ell_\infty} \\ \quad - \|\theta_{13}(t_k)\|_{\ell_\infty}\|\theta_{32}(t_k)\|_{\ell_\infty}\|\theta_{21}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{11}(t_k)\|_{\ell_\infty})\|\theta_{23}(t_k)\|_{\ell_\infty}\|\theta_{32}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{22}(t_k)\|_{\ell_\infty})\|\theta_{13}(t_k)\|_{\ell_\infty}\|\theta_{31}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{33}(t_k)\|_{\ell_\infty})\|\theta_{12}(t_k)\|_{\ell_\infty}\|\theta_{21}(t_k)\|_{\ell_\infty} > 0. \end{cases} \tag{23}$$

Figure 11 shows Δ_1, Δ_2, and Δ_3 of this example. As is shown in the figure, the stability (bounded) condition is satisfied at least in the time domain ($t_k \leq 100$).

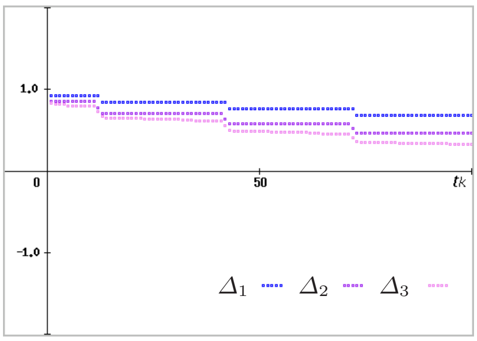


Fig. 11: The principal minors, Δ_1, Δ_2, and Δ_3.

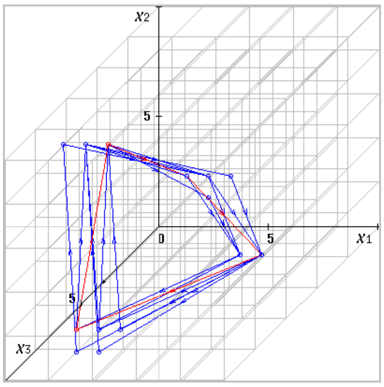


Fig. 12: State trajectory when $x_1(0) = 2.0$, $x_2(0) = 3.0$ and $x_3(0) = 1.0$.

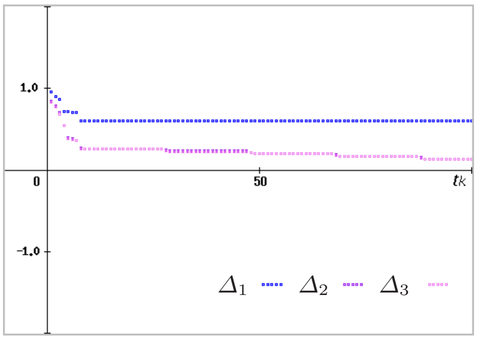


Fig. 13: The principal minors, Δ_1, Δ_2, and Δ_3.

Example 2. Next, consider the following third-order system:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k + \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}_k \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k . \tag{24}$$

The structure matrix is given below:

$$\begin{cases} \mathcal{P} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & \mathcal{P}^2 = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \\ \mathcal{P}^3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, & \mathcal{P}^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{cases}$$

In this example, the nominal system contains a periodic mode with $p = 4$. The event-driven signals e_1, e_2, and e_3 are the same as shown in Fig. 8. Figure 12 shows the state trajectory of this example. Here, we will apply Theorem 1 to this problem. Figure 13 shows Δ_1, Δ_2, and Δ_3 of this example. As shown in the figure, the stability condition is satisfied at least in the time domain ($t_k \leq 100$).

9 Conclusions

The stability of event-driven control systems has been studied using multiple metrics on lattice coordinates. First, the lattice concept (and theory) based on ordered sets was reviewed in general and the graphical representations of event-driven discrete systems were compared. Then, the (technological) state transitions were considered on the (integer) lattice coordinates. Lastly, the stability of such event-driven discrete systems was analyzed using multiple metrics and simultaneous linear inequalities. Although event-based control systems have been already studied[15)], the author thinks that those problems correspond to the traditional analysis and design of nonlinear feedback systems with a switching element.

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