Stability Analysis of Discrete Event Control Systems Using Structural Nonnegative Matrices

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Abstract There are many finite-state and event-driven types of discrete systems, e.g., manufacturing processes, industry and welfare robots, and networked control systems, and so forth. However, the analysis and design of such discrete control systems have not been established, because those systems have severe nonlinear characteristics and do not respond continuously in time. In this paper, the stability (and security) of discrete-event control systems are analyzed based on multiple metrics and structural nonnegative matrices. A stability condition is derived based on the concept of nonnegative inverse matrices (i.e., so-called M-matrices). Numerical examples are shown to clarify the stability and boundedness of discrete event systems.

Key Words: Discrete event systems, Digitally networked systems, Multiple metrics, Nonnegative inverse matrices, Relative stability

1 Introduction

There are many finite state and event-driven types of discrete systems, e.g., manufacturing systems, industry and welfare robots, computer networks, and so on. However, the analysis and design of such discrete dynamic systems have not been established [1, 2, 3], because those systems have severe nonlinear characteristics and do not respond continuously in time. In this paper, the stability of discrete-event control systems are analyzed using multiple metrics, simultaneous inequalities, and nonnegative matrices [4, 5]. As a result, a stability condition is derived based on the concept of nonnegative inverse matrices (so-called M-matrices [6]).

2 Discrete-Event Control Systems and State Traces

In general, finite-state and discrete-event systems can be written as:

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{f}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)) \quad (1)$$
$$k \in \mathbb{N} := \{0, 1, 2, \ldots, N\},$$

where
$$\boldsymbol{x}(t_k) \in \mathbb{Z}^n, \quad \boldsymbol{e}(t_k) \in \mathbb{Z}^m, \boldsymbol{f}: \ \mathbb{Z}^n \times \mathbb{Z}^m \to \mathbb{Z}^n.$$

Although $\mathbb{Z}$ is considered a finite set of integers, we define the following expression with resolution value γ as follows:
$$\mathbb{Z}_\gamma := \{-N\gamma, \cdots, -2\gamma, -\gamma, 0, \gamma, 2\gamma, \cdots, N\gamma\}.$$

Here, $\mathbb{Z}_{\gamma+}$ denotes its positive area, and $\mathbb{Z}_1 = \mathbb{Z}$, $\mathbb{Z}_+ = \mathbb{N}$. In the above expression (2), time-driven

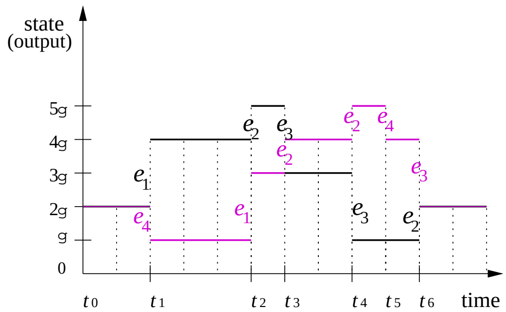


Fig. 1: State trajectories of a discrete event system.

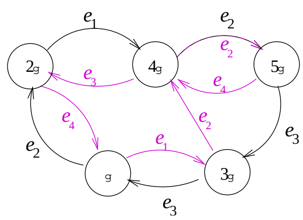


Fig. 2: State transition graph for a DES.

types of discrete systems are considered in principle, although input sequence $\boldsymbol{u}(t_k)$ can correspond to an event-sequence vector. Figure 1 shows an example of state (or output) trajectory for an event-sequence $\{e_1e_2e_3e_3e_2\cdots\}$ (& $e_4e_1e_2e_2e_4e_3\cdots$). As is obvious, it can be considered that the event sequence corresponds to time sequence $\{t_1t_2t_3t_4t_5t_6\cdots\}$. However, the causality relationship between them will be opposite. In addition, they are not one-to-one correspondence.

In order to study the relative stability problem [8, 9], we consider the following semi-linear discrete systems.

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{\Phi}(t_{k+1}, t_k)\boldsymbol{x}(t_k) + \boldsymbol{f}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)). \quad (2)$$

In this paper, the event-driven term $\boldsymbol{f}(\cdot,\cdot)$ is simplified as

$$\varepsilon_{ij}(t_k) = \frac{f_i(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k))}{x_j(t_k)} \in \mathbb{R}. \qquad (3)$$

As for a matrix expresssion, the following can be given:

$$\boldsymbol{\mathcal{E}}(t_k) = \begin{bmatrix} \varepsilon_{11}(t_k) & \dots & \varepsilon_{1n}(t_k) \\ \vdots & \ddots & \vdots \\ \varepsilon_{n1}(t_k) & \dots & \varepsilon_{nn}(t_k) \end{bmatrix}. \qquad (4)$$

Based on the above premise, (2) can be written as:

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{\Phi}(t_{k+1}, t_k)\boldsymbol{x}(t_k) + \boldsymbol{\mathcal{E}}(t_k))\boldsymbol{x}(t_k). \qquad (5)$$

Then, it can be expanded as follows:

$$\boldsymbol{x}(t_k) = \boldsymbol{\Phi}(t_k, t_0)\boldsymbol{x}(t_0) + \sum_{l=1}^{k} \boldsymbol{\Phi}(t_k, t_l)\boldsymbol{\mathcal{E}}(t_{l-1})\boldsymbol{x}(t_{l-1}). \qquad (6)$$

If the transition matrix $\boldsymbol{\Phi}(t_{k+1}, t_k)$ is piecewise constant as written below:

$$\bar{\Phi}(t_k) := \boldsymbol{\Phi}(t_{k+1}, t_k) \in \mathbb{Z}^{n\times n} \ : \text{const. for } t_k \le t \le t_{k+1}, \qquad (7)$$

the nominal system is expressed as follows:

$$\boldsymbol{x}(t_k) = \left(\prod_{h=0}^{k-1} \bar{\Phi}(t_h)\right) \boldsymbol{x}(t_0).$$

Therefore, (2) can be rewritten as

$$\boldsymbol{x}(t_k) = \left(\prod_{h=0}^{k-1} \bar{\Phi}(t_h)\right) \boldsymbol{x}(t_0) + \sum_{l=1}^{k} \left(\prod_{h=l}^{k-1} \bar{\Phi}(t_h)\right) \boldsymbol{\mathcal{E}}(t_{l-1})\boldsymbol{x}(t_{l-1}).$$

In the case of "packet losses" and/or "unexpected delays", the above expression will also be applied to the stability and security of networked control systems [10, 11].

3 Nonnegative Matrices and Graphs

Considering the structural properties of systems, we define the following nonnegative constant matrix:

$$\boldsymbol{\mathcal{P}} := \bar{\Phi}(t_k) \in \mathbb{Z}_+^{n\times n}, \quad \forall k \in \mathbb{Z}_;. \qquad (8)$$

When we simplify it only with respect to the system structure, the transition matrix can be written as:

$$\boldsymbol{\mathcal{P}} \in \mathbb{I}_+ \subseteq \mathbb{Z}_+^{n\times n}, \qquad \mathbb{I}_+ := \{0, 1\}. \qquad (9)$$

Here, a matrix each of whose enries is either 0 or 1 is called a **(0,1)-matrix** [5].

Directed graphs and (0,1)-matrices The pair

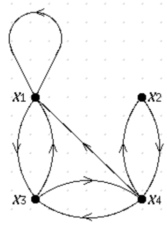


Fig. 3: Strongly connected graph.

$\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ is called a *directed graph*. Here, the elements of $\boldsymbol{V}$ are its *vertices*, and the elements of $\boldsymbol{E}$ are the *arcs* (or *edges*) of $\boldsymbol{G}$. The arc (i, j) is said to *join* vertex i to vertex j. A sequence of arcs $(i, t_1), (t_1, t_2), ..., (t_m, j)$ in $\boldsymbol{G}$ is called a *path* connecting i to j. The length of the path is defined to be the number m of arcs in the sequence. A path of length m connecting vertex i to itself is called a *cycle*.

Adjacency matrices The *adjacency* matrix $\boldsymbol{A}$ of a directed graph $\boldsymbol{G}$ with n vertices is the (0,1)-matrix (simply also called *connection* matrix) whose (i, j) entry is 1 if and only if (i, j) is an arc of $\boldsymbol{G}$. For example, an adjacency matrix of a directed graph in Fig. 3 is given below:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

However, when an algebraic operation is executed for a reccurent system, the above definition of adjacent matrix should be rewritten as "the (0,1)-matrix whose (i, j) entry is 1 if and only if (j, i) is an arc of $\boldsymbol{G}$". That is, the recurrent operation should be given as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{k+1} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_k, \quad k \in \mathbb{N}$$

with respect to $\boldsymbol{A}^T$. Therefore, in this paper, the above $A = \boldsymbol{A}^T$ will be used as a sturacture matrix.

Strongly connected graphs A directed graph $\boldsymbol{G}$ is said to be *strongly connected* if for any ordered pair of distict vertices, i and j, there is a path in $\boldsymbol{G}$ connecting i to j. The directed graph in Fig. 3 is strogly connected whereas the graph in Fig. 4 is not strongly connected because it does not contain any paths connecting x_1 to x_4.

Irreducible matrices and cogrediecy A matrix $\hat{A}$ is said to be *cogredient (not comparable)* to a matrix A if there exists a permutation matrix P such that $\hat{A} = P^T AP$ [4, 5]. Moreover, a matrix A is called

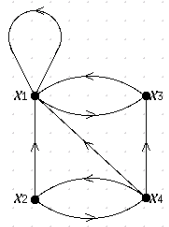


Fig. 4: Not strongly connected graph.

reducible (decomposable) if it is cogredient to

$$\hat{A} = \begin{bmatrix} C & D \\ 0 & E \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} C & 0 \\ D & E \end{bmatrix}.$$

Otherwise, A is *irreducible (indecomposable)*.

When we consider the permutation matrix (i.e., $x_3 \leftrightarrow x_4$)

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = P^T,$$

$\hat{A}$ can be obtained as follows:

$$\hat{A} = P^T A P = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

In this case, the matrix is irreducible.

On the other hand, the adjacency matrix of a graph as shown in Fig. 4 is given as

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \boldsymbol{B}^T = B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

When we consider the permutation matrix ($x_2 \leftrightarrow x_4$)

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$\hat{B}$ is obtained as

$$\hat{B} = P^T B P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

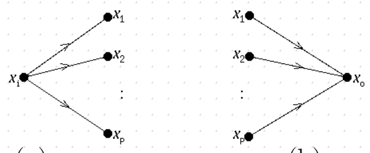
Therfore, it is clearly reducible.

Prohibition of addition in graphs Figure 5 is illustrated with respect to 'Addition' in graphs for DES. Although the additive opeartion is not affected in Fig. 5 (a), it should be prohibited in Fig. 5 (b). (In time-sharing or event-driven systems, it may be admitted in some cases.)

4 Cyclic Nominal Systems

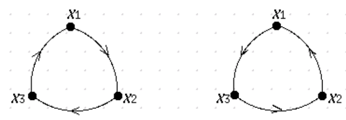
Here, we consider cyclic (periodic) nominal systems. As for third-order periodic systems, the (0,1)-matrices are given by

$$\boldsymbol{\mathcal{P}}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{\mathcal{P}}_3^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$



(a) (b)

Fig. 5: Addition-prohibited graph.



(a) (b)

Fig. 6: Directed graphs for $\boldsymbol{\mathcal{P}}_3$ and $\boldsymbol{\mathcal{P}}_3^T$.

and the directed graphs are as shown in Fig. 6 Furthermore, as for fourth-order periodic systems, the (0,1)-matrices are given by

$$\boldsymbol{\mathcal{P}}_{41} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \boldsymbol{\mathcal{P}}_{42} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \boldsymbol{\mathcal{P}}_{43} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\mathcal{P}}_{41}^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{\mathcal{P}}_{42}^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \boldsymbol{\mathcal{P}}_{43}^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and the directed graphs are as shown in Figs. 7 and 8. In general, there are $(n-1)!$ permutations with respect to n vertices (i.e., $(n-1)!$ directed graphs (n-cycle digraphs)).

In this paper, using such structural matrices, (8) can be written as follows:

$$\boldsymbol{x}(t_k) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0) + \sum_{l=1}^{k} \boldsymbol{\mathcal{P}}^{k-l} \boldsymbol{\mathcal{E}}(t_{l-1}) \boldsymbol{x}(t_{l-1}). \quad (10)$$

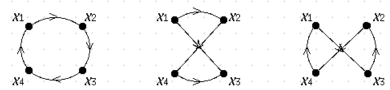
Here, consider a new type of transition matrix, i.e.,

$$\boldsymbol{\Psi}(k,l) := \boldsymbol{\mathcal{P}}^{k-l} \boldsymbol{\mathcal{E}}(t_{l-1}). \quad (11)$$

Thus, (10) can be rewritten as follows:

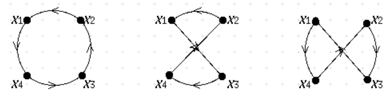
$$\boldsymbol{x}(t_k) = \bar{\boldsymbol{x}}(t_k) + \sum_{l=1}^{k} \boldsymbol{\Psi}(k,l) \boldsymbol{x}(t_{l-1}), \quad (12)$$

where $\bar{\boldsymbol{x}}(t_k) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0)$ is the nominal state response.



(a) (b) (c)

Fig. 7: Directed graphs for $\boldsymbol{\mathcal{P}}_{41}$, $\boldsymbol{\mathcal{P}}_{42}$, and $\boldsymbol{\mathcal{P}}_{43}$.



(a) (b) (c)

Fig. 8: Directed graphs for $\boldsymbol{\mathcal{P}}_{41}^T$, $\boldsymbol{\mathcal{P}}_{42}^T$, and $\boldsymbol{\mathcal{P}}_{43}^T$.

5 Multiple Metrics and Simultaneous Linear Inequalities

The metric in the state space (i.e., vector space) is usually defined by a scalar value. However, it may lead to a severe condition for the stability of some kind of nonlinear systems. Therefore, we consider the metric (i.e., norm) for each element of the state as follows:

$$\|x_i(t_k)\|_{\ell_\infty} := \sup_{1\le l\le k} |x_i(t_l)| \in \mathbb{Z}_+. \tag{13}$$

When considering multiple metrics, the following vector can be written:

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} = \begin{bmatrix} \|x_1(t_k)\|_{\ell_\infty} \\ \|x_2(t_k)\|_{\ell_\infty} \\ \vdots \\ \|x_n(t_k)\|_{\ell_\infty} \end{bmatrix} \in \mathbb{Z}_+^n. \tag{14}$$

The author's thought is that by using the above vectors and matrices the simultaneous equation (12) with uncertain terms should be covered by the following simultaneous inequality [1]:

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty} + \Big\| \sum_{l=1}^{k} \boldsymbol{\Psi}(k,l)\boldsymbol{x}(t_{l-1}) \Big\|_{\ell_\infty}. \tag{15}$$

Here, we define the following nonnegative matrix:

$$\|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} := \begin{bmatrix} \|\theta_{11}(t_k)\|_{\ell_\infty} & \dots & \|\theta_{1n}(t_k)\|_{\ell_\infty} \\ \vdots & \ddots & \vdots \\ \|\theta_{n1}(t_k)\|_{\ell_\infty} & \dots & \|\theta_{nn}(t_k)\|_{\ell_\infty} \end{bmatrix},$$

where,

$$\|\theta_{ij}(t_k)\|_{\ell_\infty} = \Big\| \sum_{l=1}^{k} \psi_{ij}(k,l)x_j(t_{l-1}) \Big\|_{\ell_\infty} / \|x_j(t_k)\|_{\ell_\infty}$$
$$\|\theta_{ij}(t_k)\|_{\ell_\infty} \in \mathbb{R}_+ \quad i,j = 1,2,\cdots,n.$$

Therefore, inequality (15) is written as

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty} + \|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} \cdot \|x(t_k)\|_{\ell_\infty}. \tag{16}$$

[1] Inequality symbols for matrices and vectors are based on [7]

Moreover, it can be given as

$$\Big(\boldsymbol{I} - \big\|\boldsymbol{\Theta}(t_k)\big\|_{\ell_\infty}\Big)\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty}. \tag{17}$$

Here, we write the above inequality as follows:

$$\big(\boldsymbol{I} - \boldsymbol{C}\big)\boldsymbol{X} \leqq \boldsymbol{Y}, \tag{18}$$

where $\boldsymbol{C} \geqq \boldsymbol{0}$, $\boldsymbol{X} \geqq \boldsymbol{0}$, and $\boldsymbol{Y} \geqq \boldsymbol{0}$ correspond to $\|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty}$, $\|\boldsymbol{x}(t_k)\|_{\ell_\infty}$, and $\|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty}$, respectively.

6 Nonnegative Inverse Matrices

Consider the following square matrix: [4]

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & a_{nn} \end{bmatrix}, \tag{19}$$

where $a_{ij} \ge 0$ $(i,j = 1,2,\cdots,n)$. It is said that the most common situation in the biological, physical, and social sciences is where the matrix $\boldsymbol{A}$ has nonpositive off-diagonal and nonnegative diagonal entries [4]. Of course, it can simply be written as $\boldsymbol{A} = [a_{i,j}]$. However, in this paper, the expression of (19) will be used in order to clarify the sign of each element.

If we apply $\boldsymbol{A}$ to $\boldsymbol{I} - \boldsymbol{C}$ in (18), with respect to

$$\boldsymbol{C} = \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \dots \\ c_{n1} & \dots & c_{nn} \end{bmatrix}$$

$1-c_{ii} \ge 0$ and $c_{ij} \ge 0$ $(i \ne j)$ are obtained, though $1-c_{ii} > 0$ is considered in the engineering applications.

When $n = 3$, a simultaneous equation,

$$\begin{bmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leqq \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}. \tag{20}$$

can be defined. By using an elimination method, the following equation is obtained based on an upper triangle matrix [9, 12].

$$\begin{bmatrix} a_{11}^{(1)} & -a_{12}^{(1)} & -a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & -a_{23}^{(2)} \\ 0 & 0 & a_{33}^{(3)} \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} \leqq \begin{bmatrix} y_1^{(1)} \\ y_2^{(2)} \\ y_3^{(3)} \end{bmatrix}, \tag{21}$$

where $x_1^{(1)} = x_1$, $x_2^{(1)} = x_2$, $x_3^{(1)} = x_3$, $a_{11}^{(1)} = a_{11}$, $a_{12}^{(1)} = a_{12}$, $a_{13}^{(1)} = a_{13}$. and thus

$$a_{22}^{(2)} = \frac{1}{a_{11}^{(1)}} \begin{vmatrix} a_{11}^{(1)} & -a_{12}^{(1)} \\ -a_{21}^{(1)} & a_{22}^{(1)} \end{vmatrix}, \quad a_{23}^{(2)} = -\frac{1}{a_{11}^{(1)}} \begin{vmatrix} a_{11}^{(1)} & -a_{13}^{(1)} \\ -a_{21}^{(1)} & -a_{23}^{(1)} \end{vmatrix}$$

$$a_{32}^{(2)} = -\frac{1}{a_{11}^{(1)}} \begin{vmatrix} a_{11}^{(1)} & -a_{12}^{(1)} \\ -a_{31}^{(1)} & -a_{32}^{(1)} \end{vmatrix}, \quad a_{33}^{(2)} = \frac{1}{a_{11}^{(1)}} \begin{vmatrix} a_{11}^{(1)} & -a_{13}^{(1)} \\ -a_{31}^{(1)} & a_{33}^{(1)} \end{vmatrix}$$

$$a_{33}^{(3)} = \frac{1}{a_{22}^{(2)}} \begin{vmatrix} a_{22}^{(2)} & -a_{23}^{(2)} \\ -a_{32}^{(2)} & a_{33}^{(2)} \end{vmatrix}.$$

Therefore, the right side of (17) is given by

$$y_1^{(1)} = y_1,\ y_2^{(2)} = y_2^{(1)} + \frac{a_{21}^{(1)}}{a_{11}^{(1)}} y_1^{(1)},\ y_3^{(3)} = y_3^{(2)} + \frac{a_{32}^{(2)}}{a_{22}^{(2)}} y_2^{(2)}.$$

and thus if $a_{11}^{(1)} > 0$, $a_{22}^{(2)} > 0$, and $a_{33}^{(3)} > 0$, the following inequalities can be obtained:

$$\begin{cases} x_3^{(1)} \le \frac{1}{a_{33}^{(3)}} y_3^{(3)} < \infty \\ x_2^{(1)} \le \frac{1}{a_{22}^{(2)}} \left(y_2^{(2)} + a_{23}^{(2)} x_3^{(2)} \right) < \infty \\ x_1^{(1)} \le \frac{1}{a_{11}^{(1)}} \left(y_1^{(1)} + a_{12}^{(1)} x_2^{(2)} + a_{13}^{(1)} x_3^{(3)} \right) < \infty. \end{cases} \quad (22)$$

It should be noted that $a_{ii}^{(i)} > 0$ $(i = 1, 2, 3)$ is rewritten as

$$\begin{cases} a_{11}^{(1)} = \Delta_1 = a_{11} > 0 \\ a_{22}^{(2)} = \frac{\Delta_2}{\Delta_1} = \begin{vmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{vmatrix} \big/ a_{11} > 0 \\ a_{33}^{(3)} = \frac{\Delta_3}{\Delta_2} = \begin{vmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{vmatrix} \big/ \begin{vmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{vmatrix} > 0 \end{cases} \quad (23)$$

Thus, from (18), (17) and (23) the stability condition is given below:

$$\begin{cases} \Delta_1 = 1 - \|\theta_{11}(t_k)\|_{\ell_\infty} > 0 \\ \Delta_2 = (1 - \|\theta_{11}(t_k)\|_{\ell_\infty})(1 - \|\theta_{22}(t_k)\|_{\ell_\infty}) \\ \quad - \|\theta_{12}(t_k)\|_{\ell_\infty} \|\theta_{21}(t_k)\|_{\ell_\infty} > 0 \\ \Delta_3 = (1 - \|\theta_{11}(t_k)\|_{\ell_\infty})(1 - \|\theta_{22}(t_k)\|_{\ell_\infty})(1 - \|\theta_{33}(t_k)\|_{\ell_\infty}) \\ \quad - \|\theta_{12}(t_k)\|_{\ell_\infty} \|\theta_{23}(t_k)\|_{\ell_\infty} \|\theta_{31}(t_k)\|_{\ell_\infty} \\ \quad - \|\theta_{13}(t_k)\|_{\ell_\infty} \|\theta_{32}(t_k)\|_{\ell_\infty} \|\theta_{21}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{11}(t_k)\|_{\ell_\infty}) \|\theta_{23}(t_k)\|_{\ell_\infty} \|\theta_{32}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{22}(t_k)\|_{\ell_\infty}) \|\theta_{13}(t_k)\|_{\ell_\infty} \|\theta_{31}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{33}(t_k)\|_{\ell_\infty}) \|\theta_{12}(t_k)\|_{\ell_\infty} \|\theta_{21}(t_k)\|_{\ell_\infty} > 0. \end{cases} \quad (24)$$

7 Numerical Examples

Example 1. Consider the following recurrent third-order system disturbed by some uncertain event-signals:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k + \begin{bmatrix} 0 & 0 & e_1 \\ e_2 & 0 & 0 \\ 0 & e_3 & 0 \end{bmatrix}_k \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k.$$

The nominal system of this example is periodic with a period "p=3" as shown in Fig. 6(a). Here, we assume the event-driven signals e_1, e_2, and e_3 are as shown in Fig. 9. Figure 10 shows state traces from $x_1(0) = -2.0$, $x_2(0) = 2.0$, $x_3(0) = 1.0$ for $t_k \le 200$. The state trace representation in the 3D coordinates

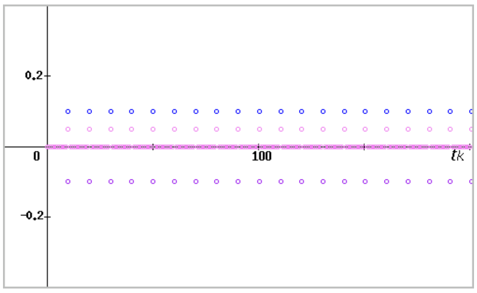


Fig. 9: Time series of event signals; peak values $e_1 = 0.1$, $e_2 = -0.1$, and $e_3 = 0.05$.

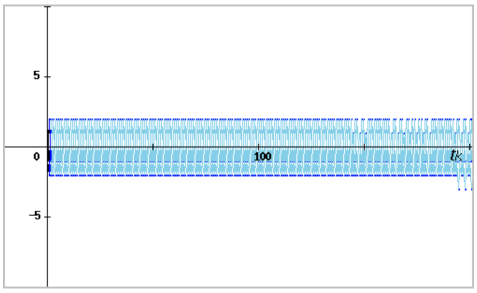


Fig. 10: State traces when $x_1(0) = -2.0$, $x_2(0) = 2.0$, and $x_3(0) = 1.0$.

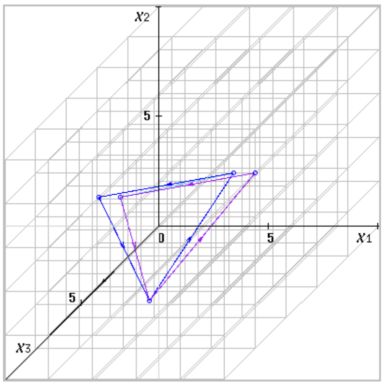


Fig. 11: A state trace in the 3D coordinates when $x_1(0) = -2.0$, $x_2(0) = 2.0$, and $x_3(0) = 1.0$ $(t_k < 300)$.

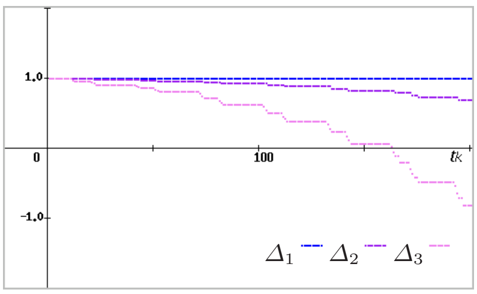


Fig. 12: Time series of Δ_1, Δ_2, and Δ_3.

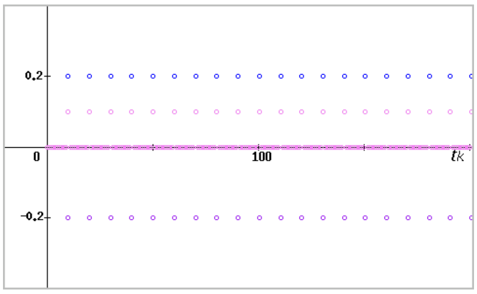


Fig. 13: Time series of event signals; peak values $e_1 = 0.2$, $e_2 = -0.2$, and $e_3 = 0.1$.

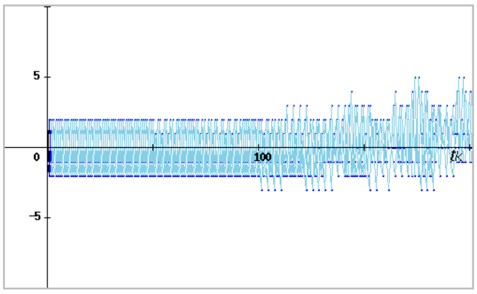


Fig. 14: State traces when $x_1(0) = -2.0$, $x_2(0) = 2.0$, and $x_3(0) = 1.0$.

is given as shown in Fig. 11. The response will be pseudo-periodic and obviously bounded (i.e., relatively stable) for $t_k \leq 200$. Figure 12 shows the time-sequences of Δ_i $(i = 1, 2, 3)$. From this figure, the stabilty (bounded) condition will be satisfied at least in the area, $t_k \leq 150$.

Example 2. When assuming the event-signals as shown in Fig. 13, the the time series of state traces become as shown im Fig. 14. Figure 15 shows the state trace in the 3D coordinates. In this case, the boundedness will not be guranteed. The time-sequences of Δ_i $(i = 1, 2, 3)$ become as shown in Fig. 16.

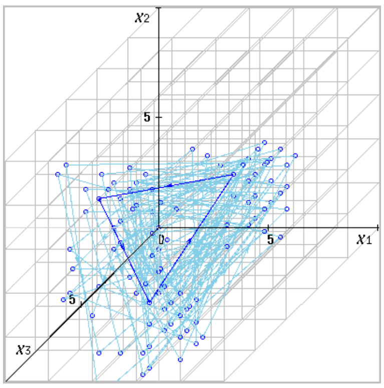


Fig. 15: A state trace in the 3D coordinates ($t_k < 200$).

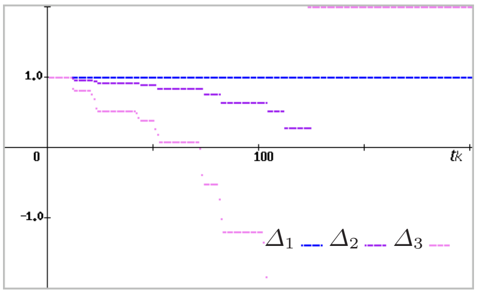


Fig. 16: Time series of Δ_1, Δ_2, and Δ_3.

8 Conclusions

The stability of discrete-event control systems has been studied using multiple metrics and simultaneous inequalities. Especially, in this paper, the structures of DES were analyzed using nonnegative (0,1)-matrices. The relationship between nonnegaive matrices and graph representations was also reviewed in general. As a result, a stability condition was derived based on the concept of nonnegative inverse matrices (so-called M-matrices).

References

[1] P. J. Ramadge, W. M. Wonham: The Control of Discrete Event Systems; *Proc. IEEE*, Vol. 77, pp. 81-98 (1989)

[2] K. M. Passino and K. L. Burgess: *The Stability Analysis of Discrete Event Systems*, John Wiley & Sons., New York (1998)

[3] C. G. Cassandras and S. Lafortune: *Introduction to Discrete Event Systems*, Springer-Verlag, Berlin (1999)

[4] A. Berman and R. J. Plemmons: Nonnegative Matrices in the Mathematical Sciences, Academic Press, London, pp. 26-62 & pp. 132-164 (1979)

[5] H. Minc: *Nonnegative Matrices*, J. Wiley & Sons, New York, pp. 68-104 (1988)

[6] A. M. Ostrowski: Note on Bounds for Some Determinants; *Duke Mathematical Journal*, Vol. 22, pp. 95-102 (1955)

[7] J. M. Ortega: *Matrix Theory; A Second Course*, Plenum Press, New York, pp. 221-224 (1987)

[8] Y. Okuyama: Relative Stability of Linear Dynamic Systems with Bounded Uncertain Coefficients; *Transaction of SICE*, Vol. 10, pp. 527-532, 1974 (in Japanese)

[9] Y. Okuyama: *Discrete Control Systems*, Springer-Verlag, London, pp. 233-236 & 152-153 (2014)

[10] J. Lunze (ed.): *Control Theory of Digitally Networked Dynamic Systems*, Springer International Publishing Switzerland pp. 171-186 (2014)

[11] Y. B. Zhao, G. P. Liu, Y. Kang, and L. Yu: *Packet-Based Control for Networked Control Systems*, Science Press, Beijin, China (2017)

[12] Y. Okuyama: On the L_2-Stability of Linear Systems with Time-Varying Parameters; *Transaction of SICE*, Vol. 3, pp. 252-259 (1967) (in Japanese).