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Discrete Model Expressions of Event-driven and Clock-driven Control Systems

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Abstract Nowadays automatic control systems in practice would be classified into two types, that is, event-driven and clock-driven ones. However, the theory of event-driven control systems has been different from the usual theory of (clock/time-driven, i.e., sampled-data) control systems. In this paper, these two types of control systems are tried to discuss in a unified theory based on the analysis of recurrent systems i.e., finite/infinite series expansion. Some numerical (asymptotic) examples are shown to illustrate the facts. The relationship between continuous and discretized control systems will be made clear from the results of this monograph.

Key Words: Discrete event systems, Recurrent dynamic systems, Quantization, Lattice coordinates Logical matrix expressions

1 Introduction

Nowadays control systems in practice is said to be classified into two types, that is, event-driven and clock-driven ones. However, the theory of event-driven control systems has been different from the usual theory of (clock/time-driven, i.e., sampled-data) control systems (so-called discrete-time control theory) [1, 2, 3, 4]. In this paper, these two types of control systems are tried to discuss in a unified theory based on the analysis of recurrent systems i.e., finite/infinite series expansion. These thoughts will be presented together with some numerical (asymptotic) examples.

2 Discrete Mathematical Models

In general, event-driven and clock-driven discrete dynamic systems can be written as the following recurrent processes

$$\boldsymbol{x}(t_{h+1}) = \boldsymbol{f}(\boldsymbol{x}(t_h), \boldsymbol{e}(t_h)) \qquad (1)$$
$$h \in \mathbb{N} := \{0, 1, 2, \ldots 1, N\},$$

where $\boldsymbol{x}(\cdot)$, $\boldsymbol{e}(\cdot)$, and $\boldsymbol{f}(\cdot,\cdot)$ are states, event-signals, and a transition function, respectively.

In this paper, each variable is considered as

$$\boldsymbol{x}(t_h) \in \mathbb{Z}_\gamma^n, \quad \boldsymbol{e}(t_h) \in \mathbb{Z}_\gamma^m, \boldsymbol{f}:\ \mathbb{Z}_\gamma^n \times \mathbb{Z}_\gamma^m \to \mathbb{Z}_\gamma^n,$$

where

$$\mathbb{Z}_\gamma := \{-N\gamma \cdots, -2\gamma, \gamma, 0, \gamma, 2\gamma, \cdots N\gamma\}$$

and the sequence of events $\boldsymbol{e}(t_h)$ is considered simply

$$t_0, t_1, \cdots, t_N \Leftrightarrow \boldsymbol{e}(t_0), \boldsymbol{e}(t_1), \cdots, \boldsymbol{e}(t_N).$$

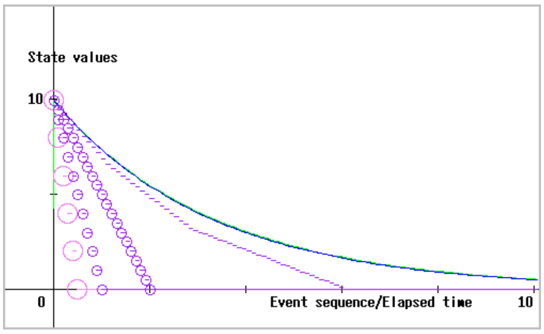

Fig. 1: Attenuating responces for a recurrent dynamic system ($\gamma = 0.1, 0.5, 1.0, 2.0$).

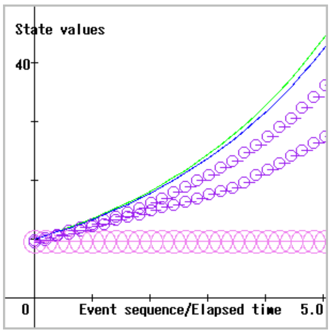

Fig. 2: Divergent responces for a recurrent dynamic system ($\gamma = 0.2, 0.55, 0.6$).

Of course, $\mathbb{Z}_1 = \mathbb{Z}$ can be defined a finite set of integers (or simply natural numbers $\mathbb{N}$). Here, $N < \infty$ and γ is the resolution of each variable[1)].

The states and events may be non-numerical (qualitative) situations in practice. However, they can be considered as ordered sets, and thus the ordered sets

[1)] Our way of thinking is that the quantity (things) in this world is finite numbers (or zeros, i.e., empty sets). The concept of infinite large and small may be an illusion in the mathematics.

will be replaced by some quantized/integral numbers. As a result, the concept of lattice theory can be applied to the analysis of qualitative dynamic systems. The multi-dimensional lattice coordinates will be presented in section 5.

When assuming $T = t_{k+1} - t_k$ to be constant in regard to the above expressions, (1) becomes a sampled-data, (i.e., clock-driven/discrete-time) system. Moreover, when the clossing terms of (1) are not considered, the expression of systems can be written as

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{A}\boldsymbol{x}(t_k). \tag{2}$$

Here, although $\boldsymbol{A}$ is a matrix expression, it should not be restricted in 'linear algebra' where additive operatons are allowed.

When using a 'continuous-like' expression, it can be given as follows:

$$\boldsymbol{x}(t_{k+1}) = (\boldsymbol{I} + \boldsymbol{A}\mathrm{d}t)\boldsymbol{x}(t_k). \tag{3}$$

Here, we can consider $\mathrm{d}t = t_{k+1} - t_k$ and $\mathrm{d}x = \boldsymbol{x}(t_{k+1}) - \boldsymbol{x}(t_k)$[2).

In the following, an elementary mathematics based on the expressions of infinie series will be presented.

3 Infinite Series Expressions

When considering a simple recurrent dynamic system, i.e.,

$$x(t_{k+1}) = (1 + \varDelta)x(t_k), \tag{4}$$

we obtain the following infinite series:

$$\begin{aligned} x(t_n) &= (1+\varDelta)^n x(t_0) \\ &= \left(1 + n\varDelta + \frac{n(n-1)}{2!}\varDelta^2 + \cdots\right) x(t_0). \end{aligned} \tag{5}$$

3.1 Attenuated Responses

In the case of an attenuated response, i.e., $\varDelta = -\sigma\mathrm{d}t$ ($\sigma > 0$),

$$\begin{aligned} x(t_n) &= (1-\sigma\mathrm{d}t)^n x(t_0) \\ &= \left(1 - n\sigma\mathrm{d}t + \frac{n(n-1)}{2!}\sigma^2\mathrm{d}t^2 + \cdots\right) x(t_0) \end{aligned} \tag{6}$$

Obviously, when considering $n \to \infty$ ($\mathrm{d}t \to 0$), the series part becomes

$$\exp(-\sigma t) = 1 - \sigma t + \frac{\sigma^2 t^2}{2!} - \frac{\sigma^3 t^3}{3!} + \cdots,$$

where $t = n\mathrm{d}t$.

Of course, when $\varDelta = \sigma\mathrm{d}t > 0$, divergent responses will be obtained as shown Fig. 2.

2) In this paper, these values are not necessarily 'small'.

3.2 Periodic Responces

Next, consider the following recurrent system,

$$\begin{bmatrix} x_1(t_{k+1}) \\ x_2(t_{k+1}) \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\omega\mathrm{d}t \\ \omega\mathrm{d}t & 0 \end{bmatrix} \right) \begin{bmatrix} x_1(t_k) \\ x_2(t_k) \end{bmatrix}, \tag{7}$$

where $\omega = 2\pi f$ (f: frequency).

We obtain the following infinite series expressions when $x_2(t_0) = 0$:

$$\begin{cases} x_1(t_n) = \left(1 - \dfrac{n(n-1)}{2!}\omega^2\mathrm{d}t^2 + \cdots\right) x_1(t_0) \\ x_2(t_n) = \left(n\omega\mathrm{d}t - \dfrac{n(n-1)(n-2)}{3!}\omega^3\mathrm{d}t^3 + \cdots\right) x_1(t_0). \end{cases} \tag{8}$$

Thus, when $n \to \infty$, the infinite series becomes,

$$\begin{cases} \cos\omega t = 1 - \dfrac{\omega^2 t^2}{2!} + \dfrac{\omega^4 t^4}{4!} - \cdots \\ \sin\omega t = \omega t - \dfrac{\omega^3 t^3}{3!} + \dfrac{\omega^5 t^5}{5!} - \cdots. \end{cases}$$

3.3 Damped Oscillations

Consider the following recurrent system,

$$\begin{bmatrix} x_1(t_{k+1}) \\ x_2(t_{k+1}) \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -\sigma\mathrm{d}t & -\omega\mathrm{d}t \\ \omega\mathrm{d}t & -\sigma\mathrm{d}t \end{bmatrix} \right) \begin{bmatrix} x_1(t_k) \\ x_2(t_k) \end{bmatrix}, \tag{9}$$

where $\sigma > 0$ and $\omega = 2\pi f > 0$. We obtain the following infinite series when $x_2(t_0) = 0$:

The above can be written as:

$$\begin{cases} x_1(t_n) = \left(1 - n\sigma\mathrm{d}t + \dfrac{n(n-1)}{2!}\sigma^{\mathrm{d}}t^2 + \cdot\right) \\ \qquad \left(1 - \dfrac{n(n-1)}{2!}\omega^2\mathrm{d}t^2 + \cdots\right) x_1(t_0) \\ x_2(t_n) = \left(1 - n\sigma\mathrm{d}t + \dfrac{n(n-1)}{2!}\sigma^2\mathrm{d}t^2 + \cdot\right) \\ \qquad \left(n\omega\mathrm{d}t - \dfrac{n(n-1)(n-2)}{3!}\omega^3\mathrm{d}t^3 + \cdots\right) x_1(t_0). \end{cases} \tag{10}$$

Thus, when $n \to \infty$, from the infinite series becomes,

$$\begin{cases} x_1(t) = \exp(-\sigma t)\cos\omega t \cdot x_1(t_0) \\ x_2(t) = \exp(-\sigma t)\sin\omega t \cdot x_1(t_0). \end{cases}$$

As describe above, dynamic systems can be annalyzed base on the finite/infinite series expressions.

4 Quantization and State Transitions

In this section, the following discretization (in other words, quantization) process is dealt with:

$$x^{\dagger} = \gamma * (\mathrm{double})(\mathrm{int})(x/\gamma) \tag{11}$$

using a C-language expression. Here, (int) and (double) denote the conversion into an integral number (a round-down discretization) and the reconversion into a double-precision real number, respectively. Baed on the above operation (11), a discretized signal $x^\dagger$ with resolution γ will be obtained.

When assuming that a set of double-precision numbers (on a PC) is $\mathbb{Z}_\delta$, discretized numbers are given as:

$$x^\dagger \in \mathbb{Z}_\gamma \subset \mathbb{Z}_\delta$$

Therefore, (11) can be written in the mathematical form as follows:

$$\frac{x^\dagger}{\gamma} = \mathcal{Q}\left(\frac{x}{\gamma}\right), \tag{12}$$

where $x \in \mathbb{Z}_\delta$, $x^\dagger \in \mathbb{Z}_\gamma$, and $\mathcal{Q} : \mathbb{Z}_\delta \to \mathbb{Z}$ based on the round-down operation, the following relation will be valid:

$$\frac{x^\dagger}{\gamma} = \mathcal{Q}\left(\frac{x^\dagger + \mathrm{d}x}{\gamma}\right).$$

Thus, when $0 < \mathrm{d}x < \gamma$, the above $x^\dagger$ is obtained.

On the other hand, when $\mathrm{d}x < 0$, some step decreasing will be obtained in order to the round-down operation.

4.1 Attenuation Process

Consider the following recurrent discretized process:

$$x(t_{h+1}) = (1 - \sigma\mathrm{d}t)x^\dagger(t_h). \tag{13}$$

Here, based on (12)

$$x^\dagger(t_h) = \gamma\mathcal{Q}(x(t_h)/\gamma).$$

Figure 1 shows the calculation results of (13) when $\sigma\mathrm{d}t = 0.03$ and $\gamma = 0.1, 0.5, 1.0, 2.0$. In this case, at least one-step decreasing will be occured because of $-\sigma\mathrm{d}t < 0$. When $\gamma = 2.0$, only five-step decreasing is occured.

In this figure 'blue' line is a curve based on the following recurrent equation.

$$x(t_{h+1}) = (1 - \sigma\mathrm{d}t)x^\dagger(t_h),$$
$$x^\dagger(t_h) \in \mathbb{Z}_\gamma, \quad \sigma\mathrm{d}t = 0.01, \quad \gamma = 0.02.$$

And 'green line' is a curve that is calculated in $\mathbb{Z}_\delta$ by using exponential function.

$$x(t) = 10\exp(-3t) \in \mathbb{Z}_\delta, \quad t = n\mathrm{d}t, \quad \mathrm{d}t = 0.01.$$

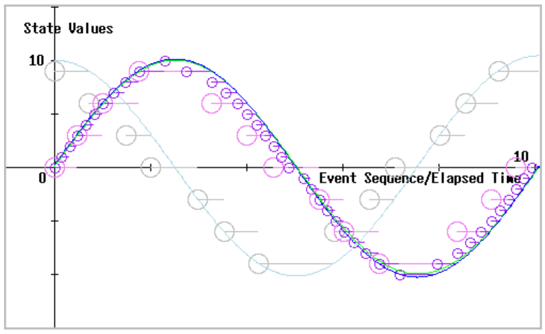

Fig. 3: Periodic responces for a recurrent dynamic system-1 ($\gamma = 1.0$ and 3.0).

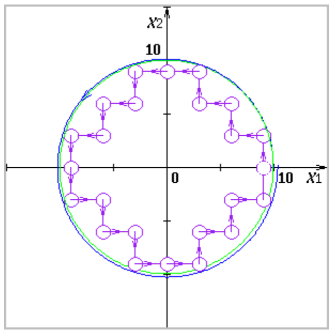

Fig. 4: Phase plane expression-1 ($\gamma = 3.0$).

On the other hand, Fig. 2 shows divergent responses when $\sigma\mathrm{d}t = 0.06$ and $\gamma = 0.2, 0.55, 0.6$, and $x(t_0) = 9.6$. In this figure, 'blue' and 'green' lines are the same cases as shown in the previous example.

4.2 Periodic Processes

Next, consider periodic systems written by (7),i.e.,

$$\begin{bmatrix} x_1(t_{h+1}) \\ x_2(t_{h+1}) \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\omega\mathrm{d}t \\ \omega\mathrm{d}t & 0 \end{bmatrix} \right) \begin{bmatrix} x_1^\dagger(t_h) \\ x_2^\dagger(t_h) \end{bmatrix}, \tag{14}$$

where $\sigma > 0$, $\omega = 2\pi f > 0$, and

$$x_i^\dagger(t_h) = \gamma\mathcal{Q}(x_i(t_h)/\gamma), \quad i = 1, 2.$$

Figure 3 shows the calculation results of (14) for $x_2(t_h)$ when $\omega\mathrm{d}t = 0.004\pi$ and $\gamma = 1.0$ and 3.0. Also in this figure, 'blue' and 'green' lines are the same cases as shown in the previous example. Moreover, the 'grey' circles and lines denote $x_1(t_h)$ and the 'light-blue' curve shows $x_1(t_h)$ for the 'blue' one.

A kind of phase plane trajectory for (x_1, x_2) is also given in Fig. 4. In the extreme case where $\gamma = 3.0$, it can be found that the response is constracted by only 24 states.

With respect to an extreme case in Fig. 5 (i.e., $\gamma = 6.0$), the response is constructed by only 8 states as shown in Fig. 6.

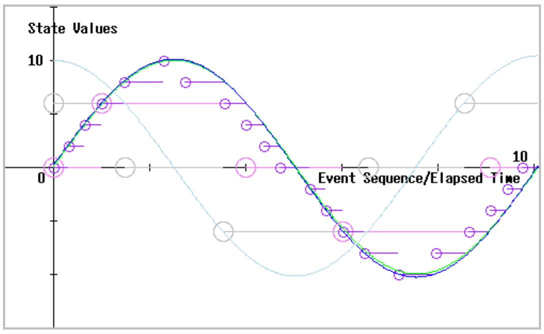

Fig. 5: Periodic responces for a recurrent system ($\gamma = 2.0$ and 6.0).

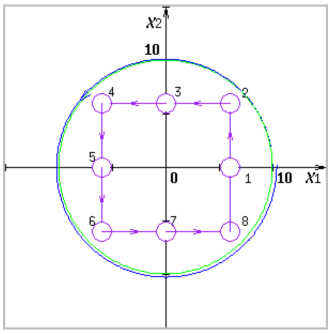

Fig. 6: Phase plane expression-2 ($\gamma = 6.0$).

4.3 Damping Oscillations

Consider the following recurrent system,

$$\begin{bmatrix} x_1(t_{k+1}) \\ x_2(t_{k+1}) \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -\sigma\mathrm{dt} & -\omega\mathrm{dt} \\ \omega\mathrm{dt} & -\sigma\mathrm{dt} \end{bmatrix} \right) \begin{bmatrix} x_1^\dagger(t_k) \\ x_2^\dagger(t_k) \end{bmatrix}, \tag{15}$$

where $\sigma > 0$, $\omega = 2\pi f > 0$, and

$$x_i^\dagger(t_h) = \gamma \mathcal{Q}(x_i(t_h)/\gamma), \qquad i = 1, 2.$$

Figure 7 shows an example of damping oscillation for step responce when $\omega\mathrm{dt} = 0.04\pi$, $\sigma\mathrm{dt} = 0.023$, and $\gamma = 2.0$ and 3.0 for $9.0 - x_1(t_h)$. (For reference, the 'lightblue' curve for $x_1(t_h)$ is drawn.)

Phase plane trajectories are given as shown in Fig. 8 for $\gamma = 3.0$ [3)].

4.4 Three Dimensional Cases

As for the above 1st and 2nd order systems, the behavior of response is easily understood from those figures, especially in Figs. 6 and 8. However, 3rd and higer order systems cannot be easily treated in those figures.

3) Phase plane trajectories are usually drawn right-turn. However in this paper they are drawn left-turn because of the unification to the 3D coordinates expression and the angle direction.

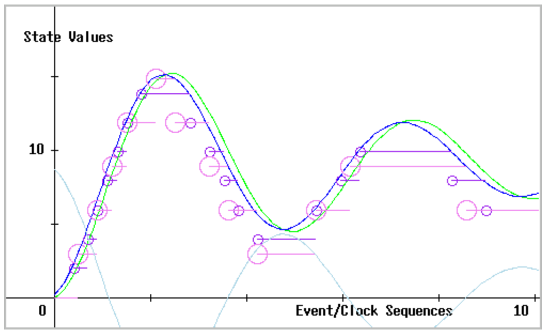

Fig. 7: An example of damping oscillation for step responces ($\gamma = 2.0$ and 3.0).

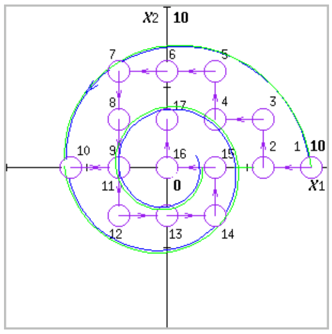

Fig. 8: Phase plane of damping case ($\gamma = 3.0$).

Here, consider the following 3rd order systems:

$$\boldsymbol{x}(t_{h+1}) = (\boldsymbol{I} + \boldsymbol{\Delta})\boldsymbol{x}^\dagger(t_h), \tag{16}$$

$$\boldsymbol{\Delta} = \begin{bmatrix} -\sigma\mathrm{dt} & -\omega\mathrm{dt} & 0 \\ \omega\mathrm{dt} & -\sigma\mathrm{dt} & 0 \\ 0 & \sigma_3\mathrm{dt} & -\sigma_3\mathrm{dt} \end{bmatrix}, \quad \boldsymbol{x}^\dagger(t_h) = \begin{bmatrix} x_1^\dagger(t_h) \\ x_2^\dagger(t_h) \\ x_3^\dagger(t_h) \end{bmatrix}$$

where $\sigma > 0$, $\omega = 2\pi f > 0$, and

$$x_i^\dagger(t_h) = \gamma \mathcal{Q}(x_i(t_h)/\gamma), \quad i = 1, 2, 3.$$

Figure 9 shows an example of damping oscillation when $\omega\mathrm{dt} = 0.04\pi$, $\sigma\mathrm{dt} = 0.311\pi$, $\sigma_3\mathrm{dt} = 0.04$ and $\gamma = 3.0$ and 1.5 for $9.0 - x_1(t_h)$. (The 'lightblue' and 'orange' lines show $x_1(t_h)$ and $x_3(t_h)$, respectively.) Phase space trajectories are given as shown in Fig. 10. Three dimensional (continuous) trajiectory may be difficult to undestand in such a figure. However, the discretized traces in Fig. 10 would be understood easily.

5 Lattice Coordinates

As shown in Fig. 10, discrete responses can be considered on a lattice coordinates. Therefore, the concept of lattice theory can be applied to the analysis of discrete dynamic systems, However, the definition of

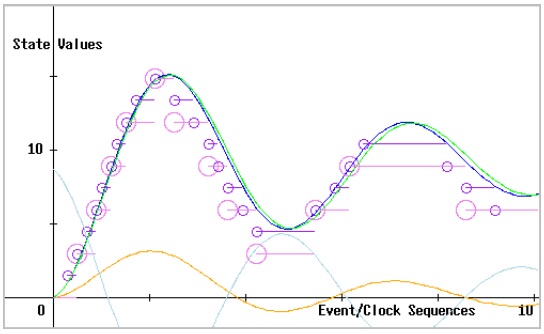

Fig. 9: An example of 3rd order system for step responses ($\gamma = 1.5$ and 3.0).

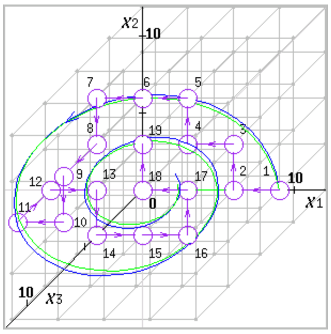

Fig. 10: Phase space responses of a 3rd order system (x_1, x_2, x_3) $(\gamma = 3.0)$.

lattices is fairly abstract expression as given below[4].

Definition of Lattices. A lattice is an ordered set L in which every pair of elements (and hence every finite subset) has an infimum (meet, $\wedge$) and a supremum (join, $\vee$). Thus, a lattice is denoted by $(L; \wedge, \vee)$, $(L; \wedge, \vee, \leq)$ or $(L; \wedge, \vee, \preceq)$ in the mathematics [5, 6, 7, 8].

Of course, the concept of lattice can be applied to simple coordinates as shown in Fig. 11. That is, $\mathbb{Z}_\gamma^2$, $\mathbb{Z}_\gamma^3$, and moreover $\mathbb{Z}_\gamma^n$ can be considered as a lattce in general. Here, the following is simply valid in Fig. 11:

$$\begin{cases} (x_1, x_2, x_3) \sqsubset (x_1 + \gamma, x_2, x_3) \\ (x_1, x_2, x_3) \sqsubset (x_1, x_2 + \gamma, x_3) \\ (x_1, x_2, x_3) \sqsubset (x_1, x_2, x_3 + \gamma), \end{cases} \tag{17}$$

where $x_i \in \mathbb{Z}_\gamma$ $(i = 1, 2, 3)$ [5].

For example, in Fig. 6 the extreme response for $\gamma = 6.0$ is given as:

$$(6, 0) \to (6, 6) \to (0, 6) \to (-6, 6) \to \cdots .$$

[4] The term 'lattice' in the mathematics is translated into 'soku' in Japanese. In English, the word 'lattice' usually means a wood structure, On the other hand, the word 'grid' corresponds to a metal structure

[5] The symbol $\sqsubset$ means 'adjacent'. These structutres can also be drawn by using Hasse diagram [9, 10].

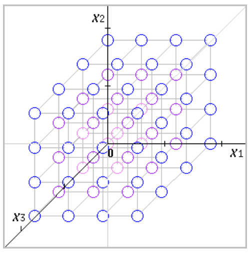

Fig. 11: Three dimensional lattice coordinates.

Obviously, the response is constructed by adjacent trasrations.

On the other hand, in Fig. 8 the exreme response for $\gamma = 3.0$ is given as:

$$(9, 0) \to (6, 0) \to (6, 3) \to (3, 3) \to (3, 6) \to \cdots .$$

Also in this case, the response is constructed by adjacent transrations. In regard to a 3rd order system shown in Fig. 10, the extreme response becomes:

$$(9, 0, 0) \to (6, 0, 0) \to (6, 3, 0) \to (3, 3, 0) \to (3, 6, 0)$$
$$\to (0, 6, 0) \to (0, 6, 3) \cdots .$$

In this example, the response is obtained by adjacent transactions,

With respect to discrete event dynamic systems, the difference between 'place' and 'token'(value) may be confused. This problem will be described below.

6 Logical Matrix Expressions

Based on the semi-tensor product (STP) and logical matrix expressions, recurrent systems can be given as follows[11, 12, 13]:

$$\boldsymbol{x}(t_{h+1}) = \boldsymbol{F}(\boldsymbol{\delta}_m^k \otimes \boldsymbol{x}(t_h) = \boldsymbol{F} \cdot \begin{bmatrix} \boldsymbol{x}_1(t_h) \\ \boldsymbol{x}_2(t_h) \\ \vdots \\ \boldsymbol{x}_m(t_h) \end{bmatrix}, \tag{18}$$

where $\boldsymbol{x}(\cdot) \in \mathbb{Z}_\gamma{}^n$, $\boldsymbol{x}_j(\cdot) \in \mathbb{Z}_\gamma{}^n$, $\boldsymbol{F} = [\boldsymbol{P}_1\ \boldsymbol{P}_2\ \cdots\ \boldsymbol{P}_{\boldsymbol{m}}]$, $\boldsymbol{P}_k \in \boldsymbol{\mathcal{P}} \subset \mathbb{Z}^{n \times n}$ $(k = 1, 2, \cdots, m)$, and $\boldsymbol{\mathcal{P}}$ is a set of $(0, 1)$-permutation matrix,

$$\boldsymbol{x}_j(t_h) = \begin{cases} \boldsymbol{x}(t_h) & \text{when } j = k \\ \boldsymbol{0} & \text{when } j \neq k. \end{cases}$$

However, there is no special meeaning in the operation. It expresses only state transitions. The operations can be written simply as

$$\boldsymbol{x}(t_{h+1}) = \boldsymbol{P}(\boldsymbol{e}(t_h)) \cdot \boldsymbol{x}(t_h), \tag{19}$$

when considering event-driven control systems[13].

For example, the state transition in Fig. 10,

$$\boldsymbol{P}(t_h) = \boldsymbol{\delta}_8[\,8\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\,]$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The relationship between the above logical matrix and the usual state transition expression, e.g., (16) will become to easy to understand, when introducing a reverse-direction idea. That is, for example, the value ('token') of place-1 is changed in order as $\hat{\boldsymbol{x}}_1(t_0)$, $\hat{\boldsymbol{x}}_2(t_0)$, $\cdots$, $\hat{\boldsymbol{x}}_p(t_0)$[6]. The processes are given below:

$$\begin{array}{ccccccc} \boldsymbol{x}(t_0) & \rightarrow & \boldsymbol{x}(t_1) & \rightarrow & \boldsymbol{x}(t_2) & \cdots & \boldsymbol{x}(t_p) \\ \| & & \| & & \| & & \| \\ \text{place-1} & & \text{place-2} & & \text{place-3} & \cdots & \text{place-p} \\ \hat{\boldsymbol{x}}_1(t_0) & & \hat{\boldsymbol{x}}_2(t_0) & & \hat{\boldsymbol{x}}_3(t_0) & \cdots & \hat{\boldsymbol{x}}_p(t_0) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \hat{\boldsymbol{x}}_1(t_1) = \hat{\boldsymbol{x}}_2(t_0) & & \hat{\boldsymbol{x}}_2(t_1) = \hat{\boldsymbol{x}}_3(t_0) & & \hat{\boldsymbol{x}}_3(t_1) = \hat{\boldsymbol{x}}_4(t_0) & \cdots & \hat{\boldsymbol{x}}_p(t_1) = \hat{\boldsymbol{x}}_1(t_0) \\ \vdots & & \vdots & & \vdots & & \vdots \\ \hat{\boldsymbol{x}}_1(t_p) = \hat{\boldsymbol{x}}_2(t_{p-1}) & & \hat{\boldsymbol{x}}_2(t_p) = \hat{\boldsymbol{x}}_3(t_{p-1}) & & \hat{\boldsymbol{x}}_3(t_p) = \hat{\boldsymbol{x}}_4(t_{p-1}) & \cdots & \hat{\boldsymbol{x}}_p(t_p) = \hat{\boldsymbol{x}}_1(t_{p-1}) \end{array}$$

when $t_p \le t_N$ and $\hat{\boldsymbol{x}}_{p+q}(t_N) = \hat{\boldsymbol{x}}_q(t_N)$ (i.e., periodic systems)

Of course, each state transition is given as:
for example, in regard to an adjacent operation on place-1 in Fig. 6,

$$\begin{bmatrix} x_1(t_1) \\ x_2(t_1) \\ x_3(t_1) \end{bmatrix} = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix}$$

and for example, in regard to an operation on place-1 in Fig. 10,

$$\begin{bmatrix} x_1(t_6) \\ x_2(t_6) \\ x_3(t_6) \end{bmatrix} = \begin{bmatrix} x_1(t_5) \\ x_2(t_5) \\ x_3(t_5) \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix}.$$

7 Conclusions

The theory of event-driven control systems has been different from the usual theory of (clock/time-driven, i.e., sampled-data) control systems. In this paper, these two types of control systems were tried to discuss in a unified theory based on the analysis of recurrent systems i.e., finite/infinite series expansion.

The analytical thoughts in this paper will be applied to 'nonlinear' dynamic systems based on 'natural' (i.e., recurrence expressions in the time domain) modeling/identification techniques.

[6] The numbers written in Figs. 6, 8, and 10 correspond to these place-numbers.

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