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Stability Analysis of Event-Driven Discrete Control Systems Based on Ordered Sets and Lattice Coordinates

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Abstract Nowadays there are many event-driven types of discrete control systems in practice, e.g., manufacturing automation systems, industrial and welfare robots, computer networked systems, and so forth. Therefore, in this paper, the (finite-time) stability of event-driven (in other words, event-based) control systems is studied. First, the lattice concept is introduced from binary relations and ordered sets in a general expression. In the lattice coordinates, the dynamics of event-driven discrete control systems is elucidated. The stability of (non-linear) discrete control systems is analyzed by using multiple metrics and simultaneous linear inequalities. Numerical examples are shown to clarify the stability and boundedness of event-driven discrete control systems.

1 Introduction

At present, there are many event-driven types of discrete systems in practice, e.g., manufacturing systems, industrial and welfare robots, computer networked systems, and so on. Therefore, in this paper, the (finite-time) stability of event-driven (in other words, event-based) control systems is studied. First, the lattice concept is introduced from binary relations and ordered sets in a general expression [1, 2, 3, 4, 5]. In the lattice coordinates, the dynamics of event-driven discrete control systems is elucidated. The stability of (non-linear) discrete control systems is analyzed by using multiple metrics and simultaneous linear inequalities. Numerical examples in the three-dimensional space (because of the visual expression) will be given to clarify the relative stability and boundedness of event-driven control systems.

2 Binary Relations and Ordered Sets

Suppose a binary relation R on a given set E satisfying the following properties:

(1) **Reflexive** $(\forall x \in E)\ (x, x) \in R$.

(2) **Anti-symmetric** $(\forall x, y \in E)$ if $(x, y) \in R$ and $(y, x) \in R$ then $x = y$.

(3) **Transitive** $(\forall x, y, z \in E)$ if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

In these descriptions, whenever $(x, y) \in R$, $x, y \in E$ are referred to as 'R-related' and often written as xRy. Here, R is called a *partial order* or, simply, an *order* relation. Moreover, the set E with the partial order is called a *partially ordered set* or, simply, '*poset*'. When specifying the relation R we will use the expression $(E; R)$.

The word 'partial' is used in defining a partially ordered set E since some of the elements of E need not be comparable. On the other hand, if any two elements of E are comparable, E is said to be *totally ordered* or *linearly ordered*, and E is called a *chain*. Incidentally,the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ of natural numbers, integers, rationals, and real numbers form chains under their usual orders[1].

An example of the quantitative relation is $\leq$ ("less than or equal") which is regarded as the *usual order*. In this case, the above properties are written as[3, 4]:

(1) **Reflexive** $(\forall x \in E)\ x \leq x$.

(2) **Anti-symmetric** $(\forall x, y \in E)$ if $x \leq y$ and $y \leq x$ then $x = y$.

(3) **Transitive** $(\forall x, y, z \in E)$ if $x \leq y$ and $y \leq z$ then $x \leq z$.

When using the expression $(E; R)$, the above relation can be specified as $(E; \leq)$. Of course, with respsect to the relation $\geq$ ("greater than or equal") the similar description can be given. Here, relations $<$ and $>$ are called 'strict' partial orders.

When considering qualitative relations, symbols $\preceq$ and $\succeq$ will be used instead of $\leq$ and $\geq$. Here, $x \preceq y$, $x \succeq y$ are read "x precedes y" and "x succeeds y", respectively[6].

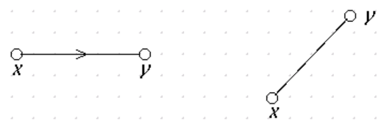
Covering relation. In an ordered set $(E; \leq)$ we say that x is covered by y (or that y covers x) if $x < y$ and there is no $z \in E$ such that $x < z < y$. We denote this by using the notation $x \sqsubset y$ [2]. Thus, points x and y that satisfy $x \sqsubset y$ are also called *adjacent*.

3 Directed Graphs and Hasse Diagrams

The adjacent points $x \sqsubset y$ can be represented by using a directed graph as shown in Fig. 1 (a). That is, we join the points (vertices) representing x and y by a line segment with an arrow (directed edge). However, instead of drawing an arrow from x to y, we sometimes place y higher than x and draw a line (without an arrow) between them as shown in Fig. 1 (b). It is then

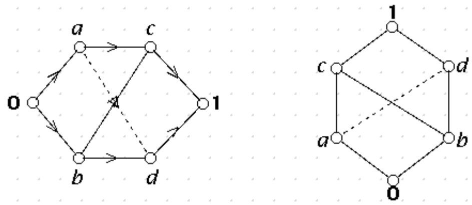
[1]The ordered set E is an antichain if $x \leq y$ in E only if $x = y$.

[2]If $x \sqsubset y$ or $x = y$, we write $x \sqsubseteq y$. These notations are used based on [4].



(a) (b)

Figure 1: Directed graph and Hasse diagram with two points.



(a) (b)

Figure 2: Directed graph and Hasse diagram with six points.

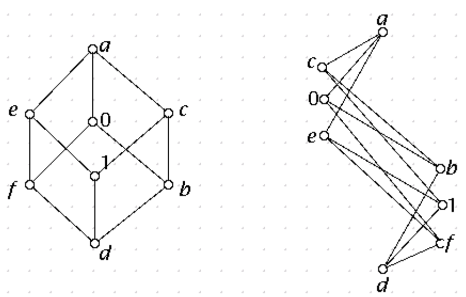
understood that an upward movement indicates succession (otherwise precession). Such a graphical representation is referred to as a *Hasse diagram*.

Figure 2 (a) is a directed graph with six points (six points lattice). In this case, by rotating 90 degrees we obtain a Hasse diagram as shown in Fig. 2 (b)[3]. Figure 3 (a) shows a Hasse diagram of cubic structure. Note that Fig. 3 (a) is isomorphic to (b) [2].

4 Lattice Structures

A lattice is an ordered set L in which every pair of elements (and hence every finite subset) has an infimum (meet, $\wedge$) and a supremum (join, $\vee$). Thus, we often denote a lattice by $(L; \wedge, \vee)$, $(L; \wedge, \vee, \leq)$ or $(L; \wedge, \vee, \preceq)$.

Another definition and properties of lattices (e.g., in [5]) are given below.



(a) (b)

Figure 3: Hasse diagram of a cubic structure.

[3]Dotted lines will be explained in the next section.

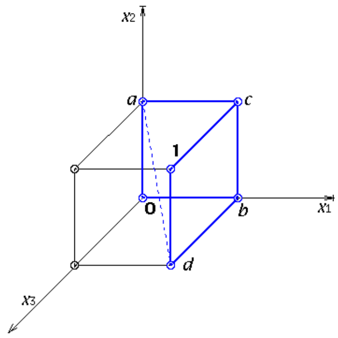
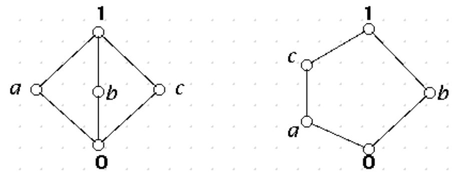


Figure 4: A basic 3D lattice.



(a) $\mathbf{M}_3$ (b) $\mathbf{N}_5$

Figure 5: Non-distributive sublattices.

Definition of Lattices [4]. Let L be an ordered set. Then L is called a *lattice* if and only if any two elements of L have a supremum and an infimum. And L is called a *colmplete lattice* if and only if any subset of L has a supremum and an infimum.

Moreover, the following special types of lattices can be defined.

Distributive Lattices. A lattice L is said to be *distributive* if it satisfies either of the following laws:

(1) $(\forall a, b, c \in L)\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(2) $(\forall a, b, c \in L)\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Note that by the principle of duality the condition (1) holds if and only if (2) holds.

Forbidden Lattices. [5] A lattice L is a distributive lattice if and only if L does not have any sublattice isomorphic to either of the two forbidden lattices $\mathbf{M}_3$ and $\mathbf{N}_5$ (five points lattice) in Fig. 5.

Modular Lattices. A lattice L is said to be *modular* if it satisfies the following law:

$$(\forall a, b, c \in L)\ a \geq c \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee c.$$

Note that the modular law is equivalent to the identity

[4]The original meaning of a "lattice" is a structure of cross laths with interstices serving as screen, door, etc (from C.O.D.). It corresponds to a "koushi' in Japanese traditional house. The word lattice in the mathematical terms is translated into Japanese as "soku" and "koushi". It should be noted that the former is with respect to **Set Theory and Topology**, e.g., [5] and the latter is with respect to **Geometry of Numbers**, e.g., [7]. In this paper, both concepts are introduced.

$$a \wedge (b \vee (a \wedge c)) = (a \wedge b) \vee (a \wedge c)$$

for all $a, b, c \in L$. This identity is obtained from the consequence of the modular law by replacing c by $a \wedge d$, which represents an arbitrary element contained in a (and the replacing d by c).

Birkoff Distributivity Criterion. A lattice L is distributive if and only if it has no sublatitce of either of the forms, $\mathbf{M}_5$, $\mathbf{N}_5$. Equivalently, L is distributive if and only if $z \wedge x = z \wedge y$ and $z \vee x = z \vee y$ imply that $x = y$.

The lattice $\mathbf{M}_3$ is called a **diamond** and the lattice $\mathbf{N}_5$ is called a **pentagon**. The following proposition is well known.

(1) A lattice L is modular if and only if it does not have a sublattice isomorphic to $\mathbf{N}_5$.

(2) A lattice L is distributive if and only if it does not have a sublattice isomorphic to either $\mathbf{N}_5$ **or** $\mathbf{M}_3$**.**

Complemented and Boolean Lattices. Let L be a lattice with $\mathbf{0}$ and $\mathbf{1}$ (i.e., a bounded lattice). We say that $b \in L$ is a *complement* of $a \in L$ if

$$a \wedge b = \mathbf{0} \quad \text{and} \quad a \vee b = \mathbf{1}.$$

A bounded lattice for which each element has a complement is called a *complemented lattice*. And a complemented 'distributive' lattice is called a *Boolean lattice*.

5 Integer Lattice Coordinates

The structure of lattice is easy to understand by using graphic representations as shown in Fig. 2 (a) and (b). In these graphs it should be noted that if vertices b and c are connected, each of diagrams does not represent a lattice since $\sup\{a, b\} = a \vee b$ cannot be determined. When we express the above diagram in a 3D Cartesian coordinates system (in other word, orthogonal system), it can be represented as shown in Fig. 4. Here, the blue lines corresponds to Fig. 2 (b).

Even if the states and events are qualitative ordered sets, we can replace them with (positive) integer numbers (i.e., a chain) as follows:

$$\begin{array}{ccccccccc} x_i^0 & \sqsubset & x_i^1 & \sqsubset & x_i^2 & \sqsubset & \cdots & \sqsubset & x_i^N \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ 0 & & 1 & & 2 & & \cdots & & N \end{array} \qquad (1)$$

$$(i = 1, 2, \cdots, n)$$

Figure 6 shows an expression of the 3D lattice coordinates. The following example of (adjacent) state series is drawn by 'purple' lines in the figure.

$$\begin{array}{ccccccc} (x_1^0, x_2^0, x_3^0) & \sqsubset & (x_1^1, x_2^1, x_3^1) & \sqsubset & (x_1^2, x_2^2, x_3^2) & & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ (0,0,0) & & (1,0,0) & & (1,1,0) & & \cdots \end{array}$$

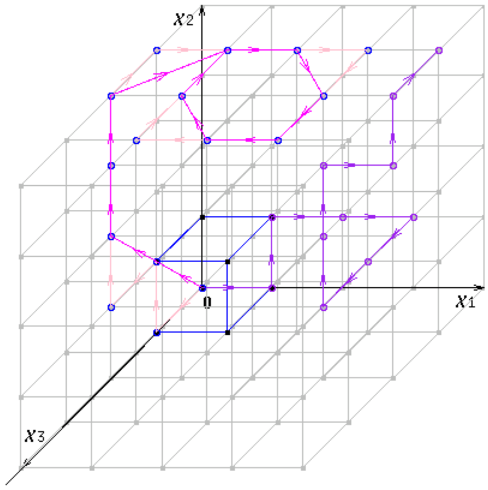


Figure 6: 3D lattice coordinates system and state traces.

$$\begin{array}{ccccc} \cdots & \sqsubset & (x_1^{11}, x_2^{11}, x_3^{11}) & \sqsubset & (x_1^{12}, x_2^{12}, x_3^{12}) \\ & & \downarrow & & \downarrow \\ \cdots & & (4,3,4) & & (4,4,4) \end{array} \qquad (2)$$

These lines and vertices construct the lattice coordinates. In Fig. 6, (1) and (2), the quadrant is restricted only to first quadrant (i.e., $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$). Of course, it can be expanded to the quadrants with negative numbers.

An example of state-trace (finally periodic trace),

$$\begin{array}{ccccccc} (x_1^0, x_2^0, x_3^0) & \prec & (x_1^1, x_2^1, x_3^1) & \prec & (x_1^2, x_2^2, x_3^2) & \prec \cdots \\ \downarrow & & \downarrow & & \downarrow & \\ (0,0,0) & & (0,1,1) & & (0,2,2) & \cdots \end{array}$$

$$\begin{array}{ccccc} \cdots (x_a^8, x_2^8, x_3^8) & \prec & (x_1^9, x_2^9, x_3^9) & \sqsubset & (x_1^{10}, x_2^{10}, x_3^{10}) \\ \downarrow & & \downarrow & & \downarrow \\ \cdots \ (2,4,3) & & (1,4,2) & & (1,4,1) \end{array}$$

is drawn with arrows by 'magenta' lines as shown in Fig. 6. Note that in this case the edges do not always connect adjacent vertices. If the state traces adjacent vertices, the edges will be drawn by 'pink' lines.

6 Discrete Systems and State Traces

In general, event-driven types of discrete systems can be written as

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{f}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)), \quad k = 0, 1, 2, \ldots. \qquad (3)$$

In order to study the relative stability problem, we consider the following semi-linear discrete systems.

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{\Phi}(t_{k+1}, t_k)\boldsymbol{x}(t_k) + \boldsymbol{f}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)). \qquad (4)$$

Here, the transition matrix $\boldsymbol{\Phi}(\cdot, \cdot)$ is considered time-invariant and written as

$$\boldsymbol{\mathcal{P}} := \boldsymbol{\Phi}(t_{k+1}, t_k) \in \mathbb{Z}^{n \times n}. \qquad (5)$$

If we simplify it only system structure, the transition matrix may be written as:

$$\boldsymbol{\mathcal{P}} \in \mathbb{I}^{n\times n} \subseteq \mathbb{Z}^{n\times n}, \quad \mathbb{I} := \{-1, 0, 1\}. \tag{6}$$

By expanding (4), the following equation is obtained:

$$\boldsymbol{x}(t_k) = \boldsymbol{\Phi}(t_k, t_0)\boldsymbol{x}(t_0) + \sum_{l=1}^{k} \boldsymbol{\Phi}(t_k, t_l)\boldsymbol{f}(\boldsymbol{x}(t_{l-1}), \boldsymbol{e}(t_{l-1})). \tag{7}$$

By using (5), it can also be written as follows:

$$\boldsymbol{x}(t_k) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0) + \sum_{l=1}^{k} \boldsymbol{\mathcal{P}}^{k-l}\boldsymbol{f}(\boldsymbol{x}(t_{l-1}, \boldsymbol{e}(t_{l-1}))). \tag{8}$$

In any case, the nominal system can be given as:

$$\bar{\boldsymbol{x}}(t_k) = \boldsymbol{\Phi}(t_k, t_0)\boldsymbol{x}(t_0) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0) \in \mathbb{Z}^n. \tag{9}$$

In this paper, the event driven function $\boldsymbol{f}$ is simplified as

$$\varepsilon_{ii}(t_k) = \frac{f_i(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k))}{x_i(t_k)} \in \mathbb{R}. \tag{10}$$

As for a matrix expression, the following can be given:

$$\boldsymbol{\mathcal{E}}(t_k) = \mathrm{diag}\{\varepsilon_{11}(t_k) \cdots \varepsilon_{nn}(t_k)\} \tag{11}$$

Based on the above premises, (8) can be written as:

$$\boldsymbol{x}(t_k) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0) + \sum_{l=1}^{k} \boldsymbol{\mathcal{P}}^{k-l}\boldsymbol{\mathcal{E}}(t_{l-1})\boldsymbol{x}(t_{l-1}). \tag{12}$$

Here, consider a new type of transition matrix, i.e.,

$$\boldsymbol{\Psi}(k, l) := \boldsymbol{\mathcal{P}}^{k-l}\boldsymbol{\mathcal{E}}(t_{l-1}).$$

Thus, (12) can be rewritten as follows:

$$\boldsymbol{x}(t_k) = \bar{\boldsymbol{x}}(t_k) + \sum_{l=1}^{k} \boldsymbol{\Psi}(k, l)\boldsymbol{x}(t_{l-1}), \tag{13}$$

where $\bar{\boldsymbol{x}}(t_k) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0)$ is the nominal state response.

7 Multiple Metrics and Inequalities

The metric in the state space (i.e., vector space) is usually defined by a scalar value. However, it may lead to a severe condition for the stability of some kind of nonlinear systems. Therefore, we consider the metric (i.e., norm) for each element of the state as follows:

$$\|x_i(t_k)\|_{\ell_\infty} := \sup_{1\le l\le k} |x_i(t_l)| \in \mathbb{Z}_+. \tag{14}$$

When considering multiple metrics, the following vector can be written:

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} = \begin{bmatrix} \|x_1(t_k)\|_{\ell_\infty} \\ \|x_2(t_k)\|_{\ell_\infty} \\ \vdots \\ \|x_n(t_k)\|_{\ell_\infty} \end{bmatrix} \in \mathbb{Z}_+^n. \tag{15}$$

Based on these considerations, the following inequalities are obtained from (13)[5]:

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty} + \left\| \sum_{l=1}^{k} \boldsymbol{\Psi}(k, l)\boldsymbol{x}(t_{l-1}) \right\|_{\ell_\infty}. \tag{16}$$

Here, we define a matrix expression with positive elements,

$$\|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} = \begin{bmatrix} \|\theta_{11}(t_k)\|_{\ell_\infty} & \dots & \|\theta_{1n}(t_k)\|_{\ell_\infty} \\ \vdots & \ddots & \vdots \\ \|\theta_{n1}(t_k)\|_{\ell_\infty} & \dots & \|\theta_{nn}(t_k)\|_{\ell_\infty} \end{bmatrix}, \tag{17}$$

where,

$$\|\theta_{ij}(t_k)\|_{\ell_\infty} := \left\| \sum_{l=1}^{k} \psi_{ij}(k, l)x_j(t_{l-1}) \right\|_{\ell_\infty} / \|x_j(t_k)\|_{\ell_\infty}$$
$$i, j = 1, 2, \cdots, n. \tag{18}$$

Therefore, inequality (16) can be written as:

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty} + \|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} \cdot \|x(t_k)\|_{\ell_\infty}. \tag{19}$$

Moreover, it can be written as follows:

$$\left(\boldsymbol{I} - \left\|\boldsymbol{\Theta}(t_k)\right\|_{\ell_\infty}\right)\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty}. \tag{20}$$

8 Relative Stability Condition

By using the above inequality expressions, the stability analysis of event-driven discrete systems is given below.

Definition. If $\|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty} \leqq \bar{\boldsymbol{X}}$ leads to $\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \boldsymbol{X}$ for all $k \in \mathbb{N}$, the discrete system is defined as (finite-time) stable in a relative sense [8]. Here, $\boldsymbol{X}$ and $\bar{\boldsymbol{X}}$ are vectors with finite (positive) components. □

Thus, the following theorem is given.

Theorem. If there exists a vector $\boldsymbol{0} \leqq \boldsymbol{X} < \infty$ by which the following inequality holds in regard to a vector $\bar{\boldsymbol{X}}$ with bounded components:

$$(\boldsymbol{I} - \|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty}\boldsymbol{X}) \leqq \bar{\boldsymbol{X}}, \tag{21}$$

the discrete system is stable in a relative sense.

In other words, the matrix of the left side of (21), i.e.,

$$\boldsymbol{A}(t_k) = \boldsymbol{I} - \|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} \tag{22}$$

is a nonnegative-inverse matrix [9] (i.e, Ostrowski's M-matrix[10, 11, 12]), the system becomes (finite-time) stable in a relative sense.

Proof. If (22) is a nonnegative-inverse matrix, (19) can be written as follows:

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \left(\boldsymbol{I} - \left\|\boldsymbol{\Theta}(t_k)\right\|_{\ell_\infty}\right)^{-1}\|\bar{\boldsymbol{x}}(t_k)\|_{\ell_\infty} < \infty. \tag{23}$$

Therefore, the relative stability of event-driven systems (12) and (13) has been proved. □

[5]Inequality symbols for vectors and matrices are based on [9]

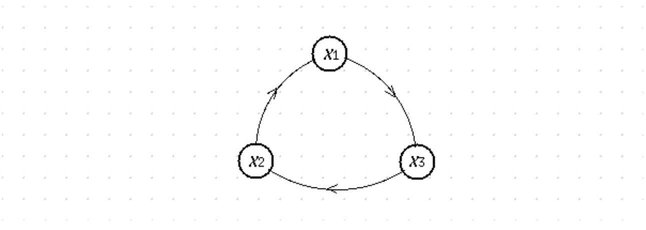


Figure 7: State transition graph of nominal system (the edge gains '1' are omitted).

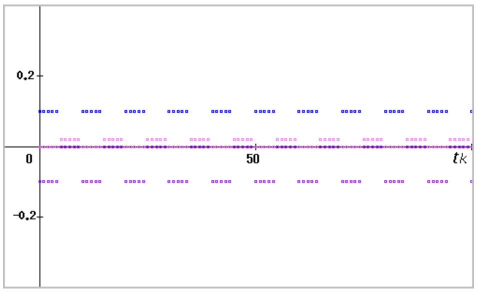


Figure 8: Time series of event signals, $e_1 = 0.1$, $e_2 = -0.1$, and $e_3 = 0.02$ (or $e_3 = 0.04$).

9 Numerical Examples

Example 1. First, consider the following third-order system with an irreducible structure matrix [13]:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k + \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}_k \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k .$$

The nominal system of this example contains a periodic mode with $p = 3$. Figure 7 shows its state transition for the irreducible case. Here we assume that event-driven signals (a kind of disturbances) e_1, e_2, and e_3 are as shown in Fig. 8. Figures. 9 and 11 are state traces based on the different event signals. The former is a stable (bounded) case when $e = 0.02$ and the latter is an unstable (divergent) pseudo-periodic case when $e = 0.04$.

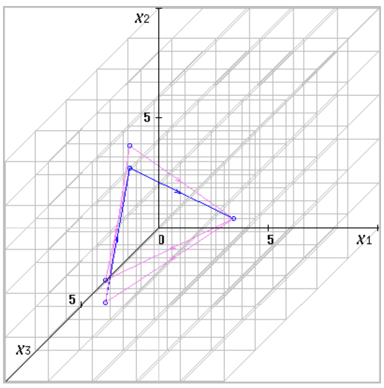


Figure 9: A state trace in the 3D coordinates when $x_1(0) = -2.0$, $x_2(0) = 2.0$, and $x_3(0) = -1.0$).

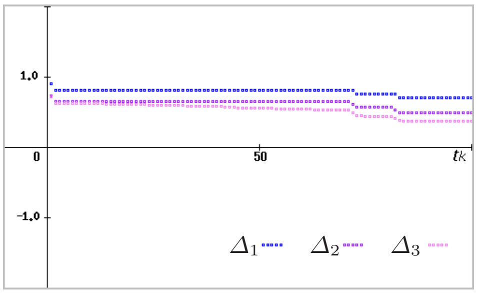


Figure 10: The principal minors, Δ_1, Δ_2, and Δ_3.

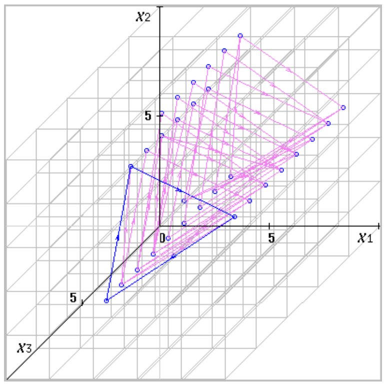


Figure 11: A state trace in the 3D coordinates when $x_1(0) = -2.0$, $x_2(0) = 2.0$, and $x_3(0) = -2.0$).

From (17), $\boldsymbol{A}(t_k)$ in **Theorem** can be written as:

$$\boldsymbol{A}(t_k) = \boldsymbol{I} - \|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} =$$

$$\begin{bmatrix} 1 - \|\theta_{11}(t_k)\|_{\ell_\infty} & -\|\theta_{12}(t_k)\|_{\ell_\infty} & -\|\theta_{13}(t_k)\|_{\ell_\infty} \\ -\|\theta_{21}(t_k)\|_{\ell_\infty} & 1 - \|\theta_{22}(t_k)\|_{\ell_\infty} & -\|\theta_{23}(t_k)\|_{\ell_\infty} \\ -\|\theta_{31}(t_k)\|_{\ell_\infty} & -\|\theta_{32}(t_k)\|_{\ell_\infty} & 1 - \|\theta_{33}(t_k)\|_{\ell_\infty} \end{bmatrix} .$$

Therefore, the stability condition is given below:

$$\begin{cases} \Delta_1 = 1 - \|\theta_{11}(t_k)\|_{\ell_\infty} > 0 \\ \Delta_2 = (1 - \|\theta_{11}(t_k)\|_{\ell_\infty})(1 - \|\theta_{22}(t_k)\|_{\ell_\infty}) \\ \quad - \|\theta_{12}(t_k)\|_{\ell_\infty}\|\theta_{21}(t_k)\|_{\ell_\infty} > 0 \\ \Delta_3 = (1 - \|\theta_{11}(t_k)\|_{\ell_\infty})(1 - \|\theta_{22}(t_k)\|_{\ell_\infty})(1 - \|\theta_{33}(t_k)\|_{\ell_\infty}) \\ \quad - \|\theta_{12}(t_k)\|_{\ell_\infty}\|\theta_{23}(t_k)\|_{\ell_\infty}\|\theta_{31}(t_k)\|_{\ell_\infty} \\ \quad - \|\theta_{13}(t_k)\|_{\ell_\infty}\|\theta_{32}(t_k)\|_{\ell_\infty}\|\theta_{21}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{11}(t_k)\|_{\ell_\infty})\|\theta_{23}(t_k)\|_{\ell_\infty}\|\theta_{32}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{22}(t_k)\|_{\ell_\infty})\|\theta_{13}(t_k)\|_{\ell_\infty}\|\theta_{31}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{33}(t_k)\|_{\ell_\infty})\|\theta_{12}(t_k)\|_{\ell_\infty}\|\theta_{21}(t_k)\|_{\ell_\infty} > 0. \end{cases} \tag{24}$$

Figure 10 shows Δ_1, Δ_2, and Δ_3 of this example (in the case of Fig. 9). As is shown in the figure, the stability (bounded) condition is satisfied at least in the time domain ($t_k \leq 271$).

Example 2. Next, consider the following third-order system with a reducible structure matrix:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k + \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}_k \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k .$$

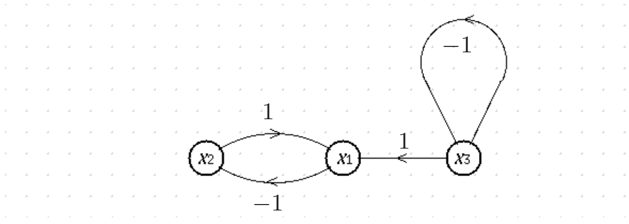


Figure 12: State transition graph of nominal system.

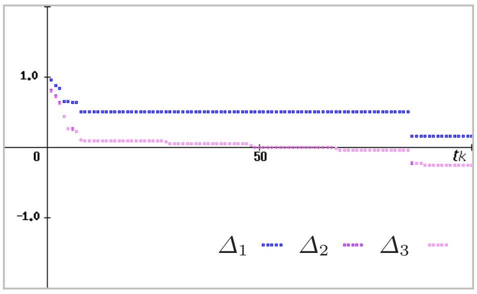


Figure 13: The principal minors, Δ_1, Δ_2, and Δ_3.

In this example, the nominal system contains a periodic mode with $p = 4$. Figure 12 shows its state transition graph for the reducible system [6]. It is assumed that the event-driven signals e_1, e_2, and e_3 are the same as Fig. 8. Here, we can apply **Theorem** to this problem. Figure 13 shows Δ_1, Δ_2, and Δ_3 of this example. As shown in the figure, the stability condition is satisfied in the time domain ($t_k < 50$). In fact, a (bounded quasi-periodic) state trace is obtained for $t_k < 200$ as shown in Fig. 14.

10 Conclusions

In this paper, the (finite-time) stability of event-driven discrete systems including qualitative problems has been studied. First, the lattice concept was introduced from binary relations and ordered sets in a general expression. As for the lattice coordinates, the dynamics of event-driven discrete control systems was examined. By using multiple metrics and simultaneous linear inequalities. the stability of (non-linear) discrete control systems was analyzed. Numerical examples clarified the relative stability and boundedness of event-driven control systems. The results of this paper will be also applied to the qualitative problems of discete event systems.

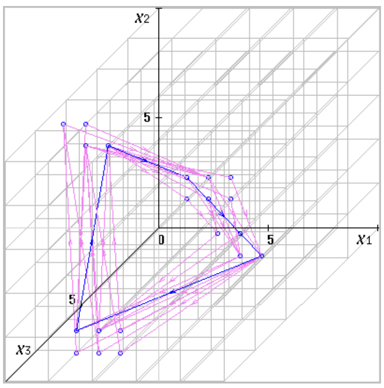


Figure 14: A state trace when $x_1(0) = 2.0$, $x_2(0) = 3.0$ and $x_3(0) = 1.0$.

[6]Since addtions are permitted in this figure, it corresponds to a signal flow graph.

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