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Periodic Oscillations and Relative Stability of Discrete Dynamical Systems on Integer Grid Coordinates

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Abstract: In the digital society, information, communication and control technologies are realized based on a packet of signals (or a finite number of data). That is, control systems are operated by discretized/quantized signals not only in the time domain but also in the spatial domain (state space). In this paper, the stability of such (nonlinear) discrete dynamical systems is analyzed based on multiple metrics and simultaneous linear inequalities on integer grid (lattice) coordinates. Numerical examples for a 3rd order discrete feedback system with saturation-type and inclined wave-type nonlinearities are presented to clarify the meaning of the relative stability and the existence of various periodic oscillations. In these examples, it can be found that there exist different limit-cycles and furthermore limit-cycles with different (longer) period.

Keywords: Digital control; finite state systems; relative stability; limit cycles; simultaneous linear inequalities

1. INTRODUCTION

In the digital society, information, communication and control technologies are realized based on a packet of signals (or a finite number of data). That is, control systems are operated by discretized/quantized signals not only in the time domain but also in the spatial domain (state space). However, the analysis and design of such discrete dynamical systems has not been established, because those systems have severe nonlinear characteristics and do not respond continuously in time. Some authors have attempted to perform the stability analysis of time-driven (continuous-value)[1, 2] and event-driven (discrete-value)[3, 4] dynamical systems. Nevertheless, many of them only discuss and define the stability (e.g., asymptotic, exponential, and Lyapunov stability) for specified discrete dynamical systems.

In this paper, the stability of such (nonlinear) discrete dynamical systems is examined using on multiple metrics and simultaneous linear inequalities [5, 6]. In consequence, non-conservative stability conditions in regard to (semi-linear) discrete dynamical systems are derived. Numerical examples for a 3rd order discrete dynamical system are presented to clarify the meaning of the relative stability and the existence of various periodic oscillations. In these examples, it can be found that there exist different limit-cycles and furthermore limit-cycles with different (longer) period.

2. GENERAL DESCRIPTION OF DISCRETE DYNAMICAL SYSTEMS

In general, when considering in the discrete state space (i.e., on integer grid coordinates), discrete dynamical systems (DDSs) can be written as follows [7, 8]:

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{f}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)) \quad (1)$$
$$\boldsymbol{x}(t_k) \in \mathbb{Z}^n, \ \ \boldsymbol{e}(t_k) \in \mathbb{Z}^m, \ \ \boldsymbol{f}: \mathbb{Z}^n \times \mathbb{Z}^m \to \mathbb{Z}^n,$$
$$k \in \mathbb{N} := \{0, 1, 2, \ldots, N\}.$$

† Yoshifumi Okuyama is the presenter of this paper.

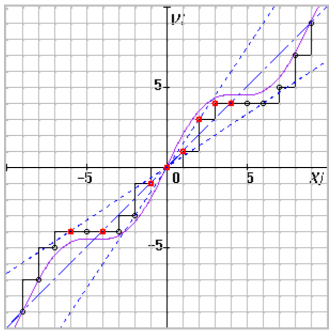

Fig. 1 Discretization of a nonlinear characteristic $\nu_i(x_j)$. Here, points ● show the traces of a response.

Although discrete dynamical systems have been studied in regard to continuous value[1, 2], in this study discrete values (i.e., discrete amplitudes) are treated. Here, $\mathbb{Z}$ is considered to be a finite set of integers. Of course, it can be extended to the following expression:

$$\mathbb{Z}_\gamma := \{-N\gamma, \cdots, -2\gamma, -\gamma, 0, \gamma, 2\gamma, \cdots, N\gamma\}, \quad (2)$$

where γ is an arbitrary resolution-value ($\gamma < 1$). However, in this paper, we will analyze the stability of discrete dynamical systems on integer grid (lattice) coordinates (i.e, $\gamma = 1$) [5, 9, 10].

In particular, the following semi-linear (relatively nonlinear) system representation will be considered:

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{\Phi}(t_{k+1}, t_k)\boldsymbol{x}(t_k) + \boldsymbol{g}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)). \quad (3)$$

Here, $\boldsymbol{g}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k))$ is a time-invariant nonlinear and discrete event vector that can be written as

$$\boldsymbol{g}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)) = \boldsymbol{\varphi}(\boldsymbol{x}(t_k)) + \boldsymbol{\eta}(\boldsymbol{x}(t_k), \boldsymbol{e}(t_k)). \quad (4)$$

In (4), the element of $\boldsymbol{\varphi}(\cdot)$ is given by a difference, e.g.,

$$\varphi_i(x_j) = \nu_i(x_j) - x_j, \quad i, j \in \{1, 2, ..., n\}, \quad (5)$$

where $\nu_i(x_j)$ is a discretized nonlinearity, for example, as shown in Fig. 1[1]. In this paper, closed-loop systems containing such a nonlinear element are considered[2].

Moreover, transition matrix $\mathbf{\Phi}(\cdot,\cdot)$ is considered time-invariant and written as

$$\boldsymbol{\mathcal{P}} := \mathbf{\Phi}(t_{k+1}, t_k) \in \mathbb{Z}^{n\times n}, \quad \forall k \in \mathbb{N}. \tag{7}$$

If we simplify it only system structure, the transition matrix may be written as:

$$\boldsymbol{\mathcal{P}} \in \mathbb{I}^{n\times n} \subseteq \mathbb{Z}^{n\times n},\ \mathbb{I} := \{-1, 0, 1\}. \tag{8}$$

In any case, the nominal system can be given as follows:

$$\mathbf{\Phi}(t_{k+1}, t_k)\boldsymbol{x}(t_k) = \boldsymbol{\mathcal{P}}\boldsymbol{x}(t_k) \in \mathbb{Z}^n. \tag{9}$$

In regard to time-invariant nominal systems, the following will be valid from (7):

(1) $\mathbf{\Phi}(t_k, t_l) = \boldsymbol{\mathcal{P}}^{k-l}$,

(2) $\mathbf{\Phi}(t_k, t_l) = \boldsymbol{\mathcal{P}}^0 = \boldsymbol{I}$.

Thus, when the nominal system is periodic, the following property can be obtained:

(3) $\mathbf{\Phi}(t_{k+p}, t_k) = \boldsymbol{\mathcal{P}}^p = \boldsymbol{I}, \quad p \in \mathbb{N}$: period.

By expanding (3), the follwing equation is obtained:

$$\boldsymbol{x}(t_k) = \mathbf{\Phi}(t_k, t_0)\boldsymbol{x}(t_0) + \sum_{l=1}^{k} \mathbf{\Phi}(t_k, t_l)\boldsymbol{\varphi}(\boldsymbol{x}(t_{l-1})). \tag{10}$$

Using (7), it can also be written as follows:

$$\boldsymbol{x}(t_k) = \boldsymbol{\mathcal{P}}^k\boldsymbol{x}(t_0) + \sum_{l=1}^{k} \boldsymbol{\mathcal{P}}^{k-l}\boldsymbol{\varphi}(\boldsymbol{x}(t_{l-1})). \tag{11}$$

In this paper, with respect to each component of time-invariant nonlinearity $\boldsymbol{\varphi}(\cdot)$, the following coefficient is considered:

$$\varepsilon_{ij}(t_k) = \frac{\varphi_i(\boldsymbol{x}(t_k))}{x_j(t_k)} \in \mathbb{R},\ \ i, j \in \{1, 2, \ldots, n\}. \tag{12}$$

Moreover, we define the following matrix:

$$\boldsymbol{\mathcal{E}}(t_k) = \begin{bmatrix} \varepsilon_{11}(t_k) & \ldots & \varepsilon_{1n}(t_k) \\ \vdots & \ddots & \vdots \\ \varepsilon_{n1}(t_k) & \ldots & \varepsilon_{nn}(t_k) \end{bmatrix}.$$

Based on the above premises, (11) can be written as:

$$\boldsymbol{x}(t_k) = \boldsymbol{\mathcal{P}}^k\boldsymbol{x}(t_0) + \sum_{l=1}^{k} \boldsymbol{\mathcal{P}}^{k-l}\boldsymbol{\mathcal{E}}(t_{l-1})\boldsymbol{x}(t_{l-1}). \tag{13}$$

[1]The discretization process, $x_j \to \nu_i^\dagger(x_j)$, is written by, for example, C-language as follows:

```
xi=dx*(double)(int)(x/dx);
ui=km0*xi+km*sin(kst*xi);
vi=dy*(double)(int)(ui/dy);
```
(6)

Here, $\texttt{x} = x_j$, $\texttt{xi} = x_j^\dagger$, $\texttt{vi} = \nu_i^\dagger(x_j)$, $\texttt{dx} = \texttt{dy} = \gamma = 1.0$, $\texttt{km0} = 1.0$, $\texttt{km} = 3.0$, and $\texttt{kst} = 0.7$ are chosen.

[2]The problems of event-driven/event-based-control [6, 11, 12]) systems will be examined in the future study.

Here, let us consider a new type of transition matrix, i.e.,

$$\mathbf{\Psi}(k, l) := \boldsymbol{\mathcal{P}}^{k-l}\boldsymbol{\mathcal{E}}(t_{l-1}).$$

As for each component, it can be expressed as

$$\mathbf{\Psi}(k, l) = \begin{bmatrix} \psi_{11}(k, l) & \ldots & \psi_{1n}(k, l) \\ \vdots & \ddots & \vdots \\ \psi_{n1}(k, l) & \ldots & \psi_{nn}(k, l) \end{bmatrix}.$$

Therefore, (10) can be rewritten as follows:

$$\boldsymbol{x}(t_k) = \boldsymbol{\mathcal{P}}^k\boldsymbol{x}(t_0) + \sum_{l=1}^{k} \mathbf{\Psi}(k, l)\boldsymbol{x}(t_{l-1}). \tag{14}$$

3. STABILITY CONDITION USING MULTIPLE METRICS

The metric in the state space is usually defined by a scalar value. However, it may lead to a severe condition for the stability of some kind of nonlinear systems (e.g., DDSs in this paper). In these cases, the multiple metrics are useful for the stability analysis [6].

Here, considering ℓ_∞ space we define the following metric (i.e., norm):

$$\|x_i(t_k)\|_{\ell_\infty} := \sup_{k\in\mathbb{N}} |x_i(t_k)| \in \mathbb{Z}_+, \quad i = 1, 2, .., n.$$

With respect to the multiple metrics, the following expression:

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} = \begin{bmatrix} \|x_1(t_k)\|_{\ell_\infty} \\ \|x_2(t_k)\|_{\ell_\infty} \\ \vdots \\ \|x_n(t_k)\|_{\ell_\infty} \end{bmatrix} \in \mathbb{Z}_+^n,$$

can be defined.

Based on the above consideration, the following inequalities,

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\boldsymbol{\mathcal{P}}^k\boldsymbol{x}(t_0)\|_{\ell_\infty} + \left\| \sum_{l=1}^{k} \mathbf{\Psi}(k, l)\boldsymbol{x}(t_{l-1}) \right\|_{\ell_\infty} \tag{15}$$

and

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\boldsymbol{\mathcal{P}}^k\|_{\ell_\infty}|\boldsymbol{x}(t_0)| + \left\| \sum_{l=1}^{k} |\mathbf{\Psi}(k, l)| \right\|_{\ell_\infty} \|\boldsymbol{x}(t_k)\|_{\ell_\infty} \tag{16}$$

are given[3]. Therefore, the following inequality holds:

$$\left(\boldsymbol{I} - \left\| \sum_{l=1}^{k} |\mathbf{\Psi}(k, l)| \right\|_{\ell_\infty}\right) \|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\boldsymbol{\mathcal{P}}^k\|_{\ell_\infty}|\boldsymbol{x}(t_0)|, \tag{17}$$

[3]The inequality symbol $\leqq$ denotes that each component of the left side vector is less than or equal to each component of the right side one. Moreover, $|\cdot|$ indicates the absolute value of each component of the vector.

where

$$\Big\| \sum_{l=1}^{k} |\boldsymbol{\Psi}(k,l)| \Big\|_{\ell_\infty} \in \mathbb{R}_+^{n\times n}.$$

In (17), it is assumed that $\boldsymbol{y}(t_k) = \boldsymbol{\mathcal{P}}^k \boldsymbol{x}(t_0)$ is bounded as shown in

$$\|\boldsymbol{y}(t_k)\|_{\ell_\infty} \leqq \|\boldsymbol{\mathcal{P}}^k\|_{\ell_\infty} \cdot |\boldsymbol{x}(t_0)| \leqq \bar{\boldsymbol{Y}} \tag{18}$$

Here, we provide the following definition in regard to the stability of discrete dynamical systems.

Definition 1.
Based on (18), if $\|\boldsymbol{y}(\boldsymbol{t_k})\|_{\ell_\infty} \leqq \bar{\boldsymbol{Y}}$ leads to $\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \bar{\boldsymbol{X}}$ for any $k \in \mathbb{N}$, the discrete dynamical system is defined as (finite-time) stable in a relative sense [13]. Here, $\bar{\boldsymbol{Y}}$ and $\bar{\boldsymbol{X}}$ are vectors of some finite (positive) numbers. □

Thus, the following theorem is obtained.

Theorem 1.
If the left side matrix in (17), i.e.,

$$\boldsymbol{\mathcal{A}} = \boldsymbol{I} - \Big\| \sum_{l=1}^{k} |\boldsymbol{\Psi}(k,l)| \Big\|_{\ell_\infty}, \quad \forall k \in \mathbb{N} \tag{19}$$

is Ostrowski's M-matrix [14] (each component of the inverse matrix of (19) is bounded and nonnegative)[4] the discrete dynamical system is (finite-time) stable in a relative sense.

Proof. If the each component of the inverse matrix of $\boldsymbol{\mathcal{A}}$ is bounded and nonnegative,

$$\|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \left(\boldsymbol{I} - \Big\| \sum_{l=1}^{k} |\boldsymbol{\Psi}(k,l)| \Big\|_{\ell_\infty} \right)^{-1} \cdot \|\boldsymbol{\mathcal{P}}^k\|_{\ell_\infty} |\boldsymbol{x}(t_0)| < \infty. \tag{20}$$

holds from (17). Then, Definition 1 is satisfied. □

Theorem 1 derived from (16) and (17) is easily applied to discrete dynamical systems, when the nominal transition matrix is asymptotic. However, when the nominal system is not asymptotic (e.g., periodic), it is difficult to apply the theorem to discrete dynamical systems. Then, we define the following matrix based on (15):

$$\boldsymbol{\Theta}(t_k) = \begin{bmatrix} \theta_{11}(t_k) & \dots & \theta_{1n}(t_k) \\ \vdots & \ddots & \vdots \\ \theta_{n1}(t_k) & \dots & \theta_{nn}(t_k) \end{bmatrix}. \tag{21}$$

Here,

$$\theta_{ij}(t_k) := \sum_{l=1}^{k} \psi_{ij}(k,l) x_j(t_{l-1}) / \|x_j(t_k)\|_{\ell_\infty} \quad i,j = 1,2,\cdots,n. \tag{22}$$

[4] $\boldsymbol{\mathcal{A}}$ in (19) is an M-matrix. That means that there exists a vector $\boldsymbol{0} \leqq \boldsymbol{X} < \infty$ with respect to an arbitrary vector, $0 \leqq \boldsymbol{Y} < \infty$ in the following equation:

$$\left(\boldsymbol{I} - \Big\| \sum_{l=1}^{k} |\boldsymbol{\Psi}(k,l)| \Big\|_{\ell_\infty} \right) \boldsymbol{X} = \boldsymbol{Y}.$$

Thus, the following theorem is given.

Theorem 2.
If the matrix based on (21), i.e.,

$$\boldsymbol{\mathcal{A}}' = \boldsymbol{I} - \|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} \tag{23}$$

is Ostrowski's M-matrix, the discrete dynamical system is (finite-time) stable in a relative sense.

Proof. With respect to the development of (19), the following inequality can be obtained using (21) and (22) instead of (17):

$$\Big(\boldsymbol{I} - \big\|\boldsymbol{\Theta}(t_k)\big\|_{\ell_\infty}\Big) \|\boldsymbol{x}(t_k)\|_{\ell_\infty} \leqq \|\boldsymbol{\mathcal{P}}^k\|_{\ell_\infty} |\boldsymbol{x}(t_0)|, \tag{24}$$

where

$$\|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} \in \mathbb{R}_+^{n\times n}, \quad \forall k \in \mathbb{N}.$$

Thus, it can be seen that if

$$\boldsymbol{\mathcal{A}}' = \boldsymbol{I} - \|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty}$$

is an M-matrix, the discrete dynamical system is relatively stable in the meaning of Definition 1.

4. NUMERICAL EXAMPLES

Example 1. In the previous paper[16], we considered 2nd order discrete dynamical systems to clarify the meaning of Theorem 2. The responses of 2nd order systems can be drawn on a (phase) plane. However, the representations will be restricted only for simple examples.

Therefore, in this paper, we will consider the following 3rd order discrete dynamical system:

$$\begin{bmatrix} x_1(t_{k+1}) \\ x_2(t_{k+1}) \\ x_3(t_{k+1}) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t_k) \\ x_2(t_k) \\ x_3(t_k) \end{bmatrix} + \begin{bmatrix} \varphi_1(x_2(t_k)) \\ 0 \\ 0 \end{bmatrix}. \tag{25}$$

and

$$\begin{bmatrix} x_1(t_{k+1}) \\ x_2(t_{k+1}) \\ x_3(t_{k+1}) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t_k) \\ x_2(t_k) \\ x_3(t_k) \end{bmatrix} + \begin{bmatrix} 0 & \varepsilon_{12}(t_k) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t_k) \\ x_2(t_k) \\ x_3(t_k) \end{bmatrix} \tag{26}$$

using expression (12). Here, $\varepsilon_{12}(t_k)$ is given by

$$\varepsilon_{12}(t_k) = \frac{\varphi_1(x_2(t_k))}{x_2(t_k)}. \tag{27}$$

In this example, with respect to the transition matrix in (25), the following operations can be obtained:

$$\boldsymbol{\mathcal{P}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \quad \boldsymbol{\mathcal{P}}^2 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\mathcal{P}}^3 = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{\mathcal{P}}^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}.$$

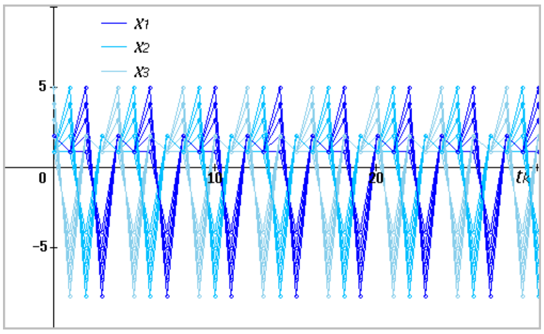

Fig. 2 Time-responses of linear periodic cases: $x_1(0) = 2.0$, $x_2(0) = 1.0$, $x_3(0) = 1.0 \sim 5.0$.

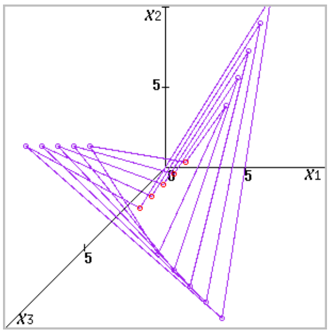

Fig. 3 Phase-space responses of linear periodic cases: $x_1(0) = 2.0$, $x_2(0) = 1.0$, $x_3(0) = 1.0 \sim 5.0$.

From the above, it can be seen the period of the nominal system is $p = 4$. Therefore, the time-responses are as shown in Fig. 2, and furthermore the responses in the 3rd order space become the set of quadrilaterals as shown in Fig. 3.

By expanding (26), the following is obtained:

$$\begin{bmatrix} x_1(t_k) \\ x_2(t_k) \\ x_3(t_k) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}^k \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \end{bmatrix}$$
$$+ \sum_{l=1}^{k} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}^{k-l} \begin{bmatrix} 0 & \varepsilon_{12}(t_{l-1}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t_{l-1}) \\ x_2(t_{l-1}) \\ x_3(t_{l-1}) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}^k \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \end{bmatrix} + \sum_{l=1}^{k} \begin{bmatrix} 0 & \psi_{12}(k,l) & 0 \\ 0 & \psi_{22}(k,l) & 0 \\ 0 & \psi_{32}(k,l) & 0 \end{bmatrix} \begin{bmatrix} x_1(t_{l-1}) \\ x_2(t_{l-1}) \\ x_3(t_{l-1}) \end{bmatrix}.$$

If we use the expression of (21) and (22), the following inequalities are given:

$$\begin{bmatrix} \|x_1(t_k)\|_{\ell_\infty} \\ \|x_2(t_k)\|_{\ell_\infty} \\ \|x_3(t_k)\|_{\ell_\infty} \end{bmatrix} \leqq \left\| \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}^k \right\|_{\ell_\infty} \cdot \begin{bmatrix} |x_1(t_0)| \\ |x_2(t_0)| \\ |x_3(t_0)| \end{bmatrix}$$
$$+ \begin{bmatrix} \|\theta_{11}(t_k)\|_{\ell_\infty} & \|\theta_{12}(t_k)\|_{\ell_\infty} & \|\theta_{13}(t_k)\|_{\ell_\infty} \\ \|\theta_{21}(t_k)\|_{\ell_\infty} & \|\theta_{22}(t_k)\|_{\ell_\infty} & \|\theta_{23}(t_k)\|_{\ell_\infty} \\ \|\theta_{31}(t_k)\|_{\ell_\infty} & \|\theta_{32}(t_K)\|_{\ell_\infty} & \|\theta_{33}(t_k)\|_{\ell_\infty} \end{bmatrix} \begin{bmatrix} \|x_1(t_k)\|_{\ell_\infty} \\ \|x_2(t_k)\|_{\ell_\infty} \\ \|x_3(t_k)\|_{\ell_\infty} \end{bmatrix}.$$

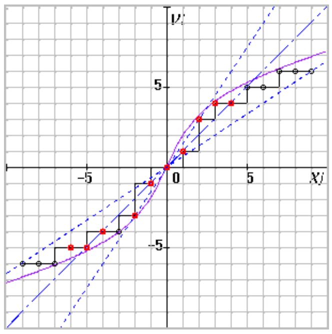

Fig. 4 Discretized nonlinear (saturation-type, arctangent curve) characteristic $\nu_1(e_2)$ for Example 4.1 and traced points $\bullet$.

Here, with respect to $i, j = 1, 2, 3$,

$$\theta_{ij}(t_k) = \sum_{l=1}^{k} \psi_{ij}(k,l) x_j(t_{l-1}) / \|x_j(t_k)\|_{\ell_\infty} \qquad (28)$$

$$\boldsymbol{I} - \|\boldsymbol{\Theta}(t_k)\|_{\ell_\infty} =$$
$$\begin{bmatrix} 1 - \|\theta_{11}(t_k)\|_{\ell_\infty} & -\|\theta_{12}(t_k)\|_{\ell_\infty} & -\|\theta_{13}(t_k)\|_{\ell_\infty} \\ -\|\theta_{21}(t_k)\|_{\ell_\infty} & 1 - \|\theta_{22}(t_k)\|_{\ell_\infty} & -\|\theta_{23}(t_k)\|_{\ell_\infty} \\ -\|\theta_{31}(t_k)\|_{\ell_\infty} & -\|\theta_{32}(t_k)\|_{\ell_\infty} & 1 - \|\theta_{33}(t_k)\|_{\ell_\infty} \end{bmatrix}$$

Therefore, the condition of M-matrix (i.e., the principal minors of $\boldsymbol{\mathcal{A}}$ is positive) is written as follows:

$$\begin{cases} \Delta_1(t_k) = 1 - \|\theta_{11}(t_k)\|_{\ell_\infty} > 0 \\ \Delta_2(t_k) = (1 - \|\theta_{11}(t_k)\|_{\ell_\infty})(1 - \|\theta_{22}(t_K)\|_{\ell_\infty}) \\ \quad - \|\theta_{12}(t_k)\|_{\ell_\infty} \|\theta_{21}(t_k)\|_{\ell_\infty} > 0 \\ \Delta_3(t_k) = (1 - \|\theta_{11}(t_k)\|_{\ell_\infty}) \\ \quad (1 - \|\theta_{22}(t_K)\|_{\ell_\infty})(1 - \|\theta_{33}(t_k)\|_{\ell_\infty}) \\ \quad - \|\theta_{12}(t_k)\|_{\ell_\infty} \|\theta_{23}(t_k)\|_{\ell_\infty} \|\theta_{31}(t_k)\|_{\ell_\infty} \\ \quad - \|\theta_{13}(t_k)\|_{\ell_\infty} \|\theta_{32}(t_k)\|_{\ell_\infty} \|\theta_{21}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{11}(t_k)\|_{\ell_\infty}) \|\theta_{23}(t_k)\|_{\ell_\infty} \|\theta_{32}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{22}(t_k)\|_{\ell_\infty}) \|\theta_{13}(t_k)\|_{\ell_\infty} \|\theta_{31}(t_k)\|_{\ell_\infty} \\ \quad - (1 - \|\theta_{33}(t_k)\|_{\ell_\infty}) \|\theta_{12}(t_k)\|_{\ell_\infty} \|\theta_{21}(t_k)\|_{\ell_\infty} > 0. \end{cases}$$

However, in this example, since the system considered here is a single-loop nonlinear feedback system, the stability condition of the above becomes only

$$\Delta_2(t_k) = \Delta_3(t_k) = 1 - \|\theta_{22}(t_k)\|_{\ell_\infty} > 0. \qquad (29)$$

4.1. Saturation-Type Nonlinearity

First, consider a saturation-type (arctangent) nonlinearity as shown in Fig. 4[5]. Fig. 5 shows the time-responses of state variables $x_1(t_k)$, $x_2(t_k)$, and $x_3(t_k)$, Fig. 6 shows

[5]The discretization process, $x_2 \to \nu_1^\dagger(x_2)$, is written by C-language as follows:

```
xi=dx*(double)(int)(x/dx);
ui=km0*xi+km*atan(kst*xi);
vi=dy*(double)(int)(ui/dy);
```
(30)

Here, x $= x_2$, xi $= x_2^\dagger$, vi $= \nu_1^\dagger(x_2)$, dx $=$ dy $= \gamma = 1.0$, km0 $= 0.3$, km $= 3.0$, and kst $= 0.6$ are chosen.

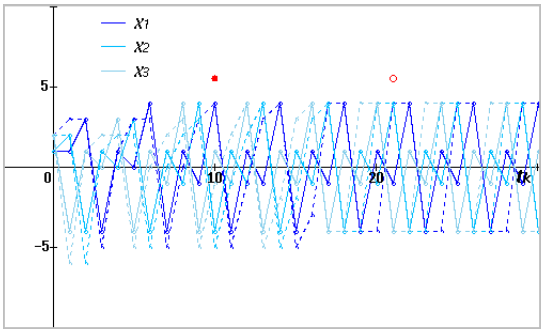

Fig. 5 Time-responses $x_1(t_k)$, $x_2(t_k)$, and $x_3(t_k)$ for Example 4.1.

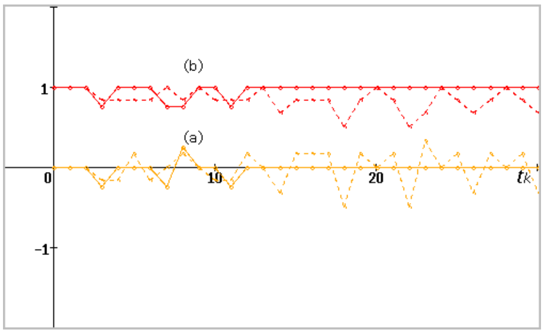

Fig. 7 (a) $\theta_{22}(t_k)$ and (b) $\Delta_1(t_k)$ for Example 4.1.

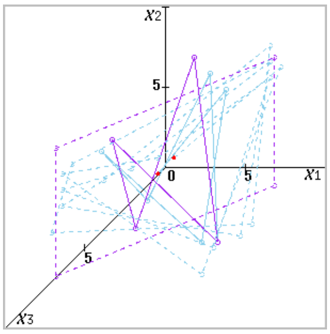

Fig. 6 Phase-space trajectories (x_1, x_2, x_3) and limit-cycles for Example 4.1.

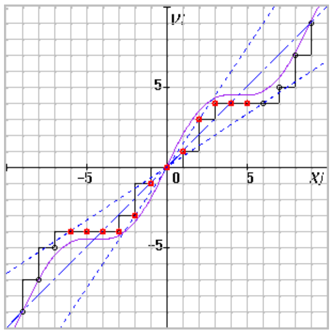

Fig. 8 Discretized nonlinear (wave-type, inclined-sine curve) characteristic $\nu_1(e_2)$ for Example 4.2 and traced points $\bullet$.

the trajectories of state variables (x_1, x_2, x_3) in the 3rd order state space, and furthermore Fig. 7 shows the time-varying processes of $\theta_{22}(t_k)$ and $\Delta_2(t_k)$.

(1) In theses figures, the solid lines show state trajectories when the initial conditions are given by $x_1(0) = 1.0$, $x_2(0) = 1.0$, and $x_3(0) = 2.0$. In this case, the trajectory approaches and results in a deformed quadrilateral with 18 steps in the 3rd order space as shown in Fig. 6. The periodic trajectory will correspond to a limit-cycle in a continuous-time system[17]. The time-response is as shown in Fig. 5 after a mark $\bullet$.

(2) On the other hand, the dotted lines show state trajectories when the initial conditions are given by $x_1(0) = 2.0$, $x_2(0) = 2.0$, and $x_3(0) = 2.0$. As is obvious in Fig. 6, in this case, the trajectory approaches and results in a quadrilateral (like a parallelogram in the 2nd order plane) with 21 steps in the 3rd order space. The time-response is as shown in Fig. 5 after a mark $\circ$.

In the above examples, the periodic trajectories (limit-cycles) obtained there depending strongly on the initial conditions. However, either of them, the relative stability will be satisfied in the meaning of Definition 1 as shown in Fig. 7.

4.2. Wave-Type Nonlinearity

Next, consider an inclined sine curve as shown in Fig. 8. In this example, `km0` $= 1.0$, `km` $= 2.9$, and `kst` $= 0.7$ are chosen in regard to (6). Fig. 9 shows the time-responses of state variables $x_1(t_k)$, $x_2(t_k)$, and $x_3(t_k)$, Fig. 10 shows the trajectories of state variables (x_1, x_2, x_3) on the 3rd order state space, and furthermore, Fig. 11 shows the time-varying processes of $\theta_{22}(t_k)$ and $\Delta_2(t_k)$.

(1) In these figures, the solid lines show state trajectories when the initial conditions are given by $x_1(0) = 2.0$, $x_2(0) = 2.0$, and $x_3(0) = 2.0$. As is shown in Fig. 10, the trajectory approaches and results in a (like) parallelogram with 13 steps in the 3rd order space. The time-response is as shown in Fig. 9 after a mark $\bullet$.

(2) On the other hand, the dotted lines show state trajectories when the initial conditions are given by $x_1(0) = 1.0$, $x_2(0) = 2.0$, and $x_3(0) = 2.0$. In this case, the trajectory approaches and results in 'like three parallelograms' (i.e., $p = 4 \times 3 = 12$) with 26 steps in the 3rd order space. The time-response is as shown in Fig. 9 after a mark $\circ$.

Also in this case, the periodic trajectories (limit-cycles) obtained there depending strongly on the initial conditions. Of course, either of them, the relative stability will be satisfied in the meaning of Definition 1 as shown in Fig. 11.

In this example, we obtained periodic responses for $p = 4$ and $p = 12$ from some different initial conditions. However, we were not able to confirm that there are periodic trajectories with other initial condition. Since nonlinear characteristics as shown in Fig. 1 and 8 can be chosen arbitrary, we may obtain periodic responses with various finite period $p < \infty$.

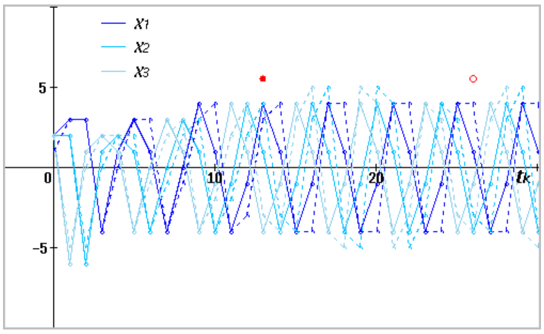

Fig. 9 Time-responses $x_1(t_k)$, $x_2(t_k)$, and $x_3(t_k)$ for Example 4.2.

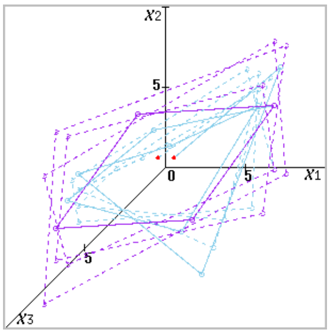

Fig. 10 Phase-space trajectories (x_1, x_2, x_3) and limit-cycles for Example 4.2.

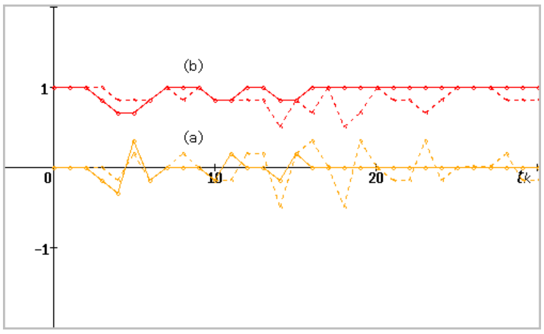

Fig. 11 (a) $\theta_{22}(t_k)$ and (b) $\Delta_2(t_k)$ trajectories for Example 4.2.

5. CONCLUSIONS

Nowadays, automatic control systems are realized by using discrete signals not only in the time domain but also in the spatial domain. In this paper, the stability of such (nonlinear) discrete dynamical systems was analyzed in a relative sense on integer grid (lattice) coordinates. Numerical examples in which arctangent and inclined-sine curves are applied to the feedback loop were shown. Especially, in this paper, a 3rd order discrete dynamical system was considered. In these examples, it was confirmed that there exist different limit-cycles and furthermore limit-cycles with extremely longer period than $p = 4$ that is the period of the nominal system.

In this study, we could not know what kind of periodic trajectory is obtained with respect to nonlinear characteristic $\nu_i(\cdot)$ and resolution-value γ. However, it is supposed that there exist periodic trajectories with various periods, because the discrete dynamical systems are considered on (finite) integer grid coordinates. Furthermore, if the resolution is minimized as $\gamma : 1.0 \to 0$, a periodic trajectory with extremely long period may be generated [18]. That will be called a situation of 'chaos'.

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